1. Let $ABCD$ be a unit square. A semicircle with diameter AB is drawn so that it lies outside of the square. If E is the midpoint of arc AB of the semicircle, what is the area of triangle CDE?

Answer: $\frac{3}{4}$

Solution: To compute the area of $\triangle CDE$, we can multiply the base of the triangle by the height of the triangle and divide by 2. Letting CD be the base, it remains to compute the perpendicular distance from E to CD. Note that the radius of the semicircle is $\frac{1}{2}$, so the perpendicular distance from E to CD must be $\frac{1}{2} + 1 = \frac{3}{2}$. Finally, $CD = 1$, so the area of $\triangle CDE$ is $\frac{1}{2} \cdot 1 \cdot \frac{3}{2} = \boxed{\frac{3}{4}}$.

2. A cat and mouse live on a house mapped out by the points $(-1, 0)$, $(-1, 2)$, $(0, 3)$, $(1, 2)$, $(1, 0)$. The cat starts at the top of the house (point $(0, 3)$) and the mouse starts at the origin $(0, 0)$. Both start running clockwise around the house at the same time. If the cat runs at 12 units a minute and the mouse at 9 units a minute, how many laps around the house will the cat run before it catches the mouse?

Answer: 2

Solution: We note that the cat and mouse start off $3 + \sqrt{2}$ units apart and the pace the cat catches up to the mouse is $12 - 9 = 3$ units a minute. Therefore, the cat will catch the mouse in $\frac{3+\sqrt{2}}{3}$ minutes. Then the cat will run $\frac{3+\sqrt{2}}{3} \times 12 = 4(3 + \sqrt{2})$ units. The perimeter of the house is $2(3 + \sqrt{2})$ units, so the cat runs $\frac{4(3+\sqrt{2})}{2(3+\sqrt{2})} = \boxed{2}$ laps.

3. In triangle ABC with $AB = 10$, let D be a point on side BC such that AD bisects $\angle BAC$. If $\frac{CD}{BD} = 2$ and the area of ABC is 50, compute the value of $\angle BAD$ in degrees.

Answer: $15°$

Solution: Since AD bisects $\angle BAC$, we have by the Angle-Bisector Theorem that $\frac{AB}{BD} = \frac{AC}{CD} \implies AC = \frac{CD}{BD} \cdot AB = 20$. Let E be the point on AC such that $BE \perp AC$. Since the area of $\triangle ABC$ is 50, we have $\frac{AC \cdot BE}{2} = 50 \implies BE = 5$. But $\triangle ABE$ is a right triangle and $AB = 2BE$, so $\triangle ABE$ must be a 30-60-90 triangle. It follows that $\angle BAC = 30°$ so $\angle BAD = \boxed{15°}$.

4. Let ω_1 and ω_2 be two circles intersecting at points P and Q. The tangent line closer to Q touches ω_1 and ω_2 at M and N respectively. If $PQ = 3$, $QN = 2$, and $MN = PN$, what is QM^2?

Answer: 6

Solution: Since MN is tangent to ω_1 at M, $\angle NMQ = \angle MPQ$. Since $MN = PN$, $\triangle MNP$ is isosceles so $\angle MPN = \angle PMN$. It follows that $\angle NPQ = \angle PMQ$. But MN is tangent to ω_2 at N, so $\angle NPQ = \angle MNQ$. Hence, $\angle MNQ = \angle PMQ$. Combining this with the fact that $\angle NMQ = \angle MPQ$, we see that $\triangle PMQ \sim \triangle MNQ$. Then $\frac{PQ}{QM} = \frac{QM}{QN}$, so $QM^2 = PQ \cdot QN = 3 \cdot 2 = \boxed{6}$.

5. The bases of a right hexagonal prism are regular hexagons of side length $s > 0$, and the prism has height h. The prism contains some water, and when it is placed on a flat surface with a hexagonal face on the bottom, the water has depth $\frac{s\sqrt{3}}{4}$. The water depth doesn't change when the prism is turned so that a rectangular face is on the bottom. Compute $\frac{h}{s}$.

Answer: $\frac{6\sqrt{3}}{5}$

Solution: When a hexagonal face is on the bottom, the volume of the water may be written as depth $(\frac{s\sqrt{3}}{4})$ times the area of the hexagonal base $(\frac{3s^2\sqrt{3}}{2})$, so the volume is $\frac{9s^3}{8}$. When a rectangular face is on the bottom, the volume of the water may be written as the cross-sectional area times the length h. The cross-section is an isosceles trapezoid with height $\frac{s\sqrt{3}}{4}$, one base of length s, and angles 120° adjacent to this base. The area of this trapezoid is $\frac{5s^2\sqrt{3}}{16}$. This can be seen by several ways. One way is to extend the trapezoid's sides past the base of length s to form an equilateral triangle, after which we may use similar triangles. Alternatively, we may notice that we can cut the cross-section into five equilateral triangles of side length $\frac{s}{2}$.

Finally, these two expressions for the volume of the water yield the equation $\frac{5s^2h\sqrt{3}}{16} = \frac{9s^3}{8}$, which can be rearranged to $\frac{h}{s} = \boxed{\frac{6\sqrt{3}}{5}}$.

6. Let the altitude of $\triangle ABC$ from A intersect the circumcircle of $\triangle ABC$ at D. Let E be a point on line AD such that $E \neq A$ and $AD = DE$. If $AB = 13$, $BC = 14$, and $AC = 15$, what is the area of quadrilateral $BDCE$?

Answer: $\mathbf{\frac{441}{4}}$

Solution: Let AD intersect BC at X. From the Pythagorean Theorem, we have that $13^2 - BX^2 = 15^2 - (14 - BX)^2$, so solving for BX yields $BX = 5$. This implies that $CX = 14 - BX = 9$ and $AX = 12$. Next, note that since $ABDC$ is cyclic, $\angle BAX = \angle DCX$ and $\angle ABX = \angle CDX$ so $\triangle ABX \sim \triangle CDX$. Then $\frac{CD}{13} = \frac{9}{12} \implies CD = \frac{39}{4}$. Also, $\frac{DX}{5} = \frac{9}{12} \implies DX = \frac{15}{4}$. By similar reasoning, $\triangle BDX \sim \triangle ACX$ so $\frac{BD}{15} = \frac{5}{12} \implies BD = \frac{25}{4}$. We also have that $\sin\angle BDE = \sin\angle BDX = \sin\angle ACX = \frac{12}{15} = \frac{4}{5}$ and $\sin\angle CDE = \sin\angle CDX = \sin\angle ABX = \frac{12}{13}$. Finally, from $AD = DE$ we have that $DE = AX + DX = 12 + \frac{15}{4} = \frac{63}{4}$. Thus,

$$\begin{aligned}
[BDCE] &= [BDE] + [CDE] \\
&= \frac{BD \cdot DE \sin\cdot\angle BDE}{2} + \frac{CD \cdot DE \cdot \sin\angle CDE}{2} \\
&= \frac{1}{2} \cdot \frac{63}{4}\left(\frac{25}{4} \cdot \frac{4}{5} + \frac{39}{4} \cdot \frac{12}{13}\right) \\
&= \frac{63}{8} \cdot (5 + 9) \\
&= \boxed{\frac{441}{4}}
\end{aligned}$$

7. Let G be the centroid of triangle ABC with $AB = 9$, $BC = 10$, and $AC = 17$. Denote D as the midpoint of BC. A line through G parallel to BC intersects AB at M and AC at N. If BG intersects CM at E and CG intersects BN at F, compute the area of triangle DEF.

Answer: $\mathbf{\frac{9}{4}}$

Solution: The centroid G cuts median AD such that $AG : GD = 2 : 1$. Since $GM || BC$, $\triangle AGM \sim \triangle ADB$. It follows that $GM : BD = 2 : 3$, and since $BC = 2BD$, $GM : BC = 1 : 3$. Furthermore, $GM || BC$ implies $\triangle GEM \sim \triangle BEC$, so $GE : BE = GM : BC = 1 : 3$.

Extend median BG so that it intersects AC at X. We know that $BG : GX = 2 : 1$, so if we let $GE = x$, we get $BE = 3x$, $BG = BE + GE = 4x$, and $GX = 2x$. It follows that $BE = EX = 3x$, so E is the midpoint of BX. But D is the midpoint of BC, so $DE || CX$. Thus, $\triangle BDE \sim \triangle BCX$, so $DE : CX = 1 : 2$, meaning that $DE : AC = 1 : 4$.

By similar reasoning, we find that $GF : CF = 1 : 3$ and $DF : AB = 1 : 4$. Combining the first ratio with $GE : BE = 1 : 3$ shows that $EF || BC$, so $\triangle GEF \sim \triangle GBC$. Hence,

$EF : BC = 1 : 4$. It follows that $\triangle DFE \sim \triangle ABC$ as the ratio of the corresponding sides is $1 : 4$.

Using Heron's Formula, the area of $\triangle ABC$ is $\sqrt{18(18-17)(18-10)(18-9)} = 36$, so the area of $\triangle DEF$ is $\frac{36}{16} = \boxed{\frac{9}{4}}$.

8. In the coordinate plane, a point A is chosen on the line $y = \frac{3}{2}x$ in the first quadrant. Two perpendicular lines l_1 and l_2 intersect at A where l_1 has slope $m > 1$. Let l_1 intersect the x-axis at B, and l_2 intersects the x and y axes at C and D, respectively. Suppose that line BD has slope $-m$ and $BD = 2$. Compute the length of CD.

Answer: $\mathbf{3 + \sqrt{13}}$

Solution: Let A' be the reflection of A across the x-axis. Since l_1 has slope m and line BD has slope $-m$, line BD is the image of l_1 when reflected across the x-axis. It follows that A' lies on line BD. Moreover, since l_2 has slope $-\frac{1}{m}$, line $A'C$ has slope $\frac{1}{m}$. Therefore, line $A'C$ is perpendicular to line BD.

Let $\angle A'DC = \theta$. We have $CD = \frac{A'D}{\cos\theta} = \frac{A'B+BD}{\cos\theta} = \frac{AB+2}{\cos\theta}$. From right triangle BAD, we have $AB = BD\sin\theta = 2\sin\theta$, so $CD = \frac{2(1+\sin\theta)}{\cos\theta}$.

Let O denote the origin. We have $\angle BOD = 90° = \angle BAD$, so $ABOD$ is cyclic. It follows that $\angle AOB = \angle ADB = \theta$. But line OA is defined by the equation $y = \frac{3}{2}x$, so $\sin\theta = \frac{3}{\sqrt{13}}$ and $\cos\theta = \frac{2}{\sqrt{13}}$. Finally, $CD = \frac{2\left(1+\frac{3}{\sqrt{13}}\right)}{\frac{2}{\sqrt{13}}} = \boxed{3+\sqrt{13}}$.

9. Let $ABCD$ be a quadrilateral with $\angle ABC = \angle CDA = 45°$, $AB = 7$, and $BD = 25$. If AC is perpendicular to CD, compute the length of BC.

Answer: $\mathbf{12\sqrt{2}}$

Solution: Let Γ be the circumcircle of $\triangle ABC$ and let line CD intersect Γ at E. Note that $ABEC$ is cyclic and $\angle ACE = 90°$ so $\angle ABE = 90°$. It follows that $\angle AEC = \angle ABC = 45°$ so $\triangle ADE$ is a 45-45-90 triangle with $AD = AE$.

Let ω be the circumcircle of $\triangle ACD$ and let line AB intersect ω at F. Note that $ACDF$ is cyclic and $\angle ACD = 90°$ so $\angle AFD = 90°$. It follows that $\angle AFC = \angle ADC = 45°$ so $\triangle BCF$ is a 45-45-90 triangle with $BC = CF$.

Observe that $\angle BEA = 90° - \angle BAE = \angle FAD$. But $AE = DA$ and $\angle AFD = 90° = \angle EBA$, so $\triangle AEB \cong \triangle DAF$. It follows that $DF = AB = 7$. Then by the Pythagorean Theorem on right triangle BDF, we have that $BF = 24$. Finally, using the 45-45-90 triangle BCF, we find that $BC = \boxed{12\sqrt{2}}$.

10. Let ABC be an acute triangle with $BC = 48$. Let M be the midpoint of BC, and let D and E be the feet of the altitudes drawn from B and C to AC and AB respectively. Let P be the intersection between the line through A parallel to BC and line DE. If $AP = 10$, compute the length of PM.

Answer: 26

Solution: Let H be the intersection of BD and CE, or in other words, the orthocenter of $\triangle ABC$. First, we show that $ADHE$ is cyclic. Note that $\angle HBC = 90° - \angle ACB$ and $\angle HCB = 90° - \angle ABC$, so

$$\angle DHE = \angle BHC = 180 - \angle HBC - \angle HCB = \angle ACB + \angle ABC$$

It follows that $\angle DHE + \angle BAC = 180°$, as desired.

Furthermore, we have that $\angle ADH = 90^\circ$, so AH is the diameter of the circumcircle of $ADHE$. But $AP||BC$ and $AH \perp BC$, so $\angle PAH = 90^\circ$. It follows that PA is tangent to the circumcircle of $ADHE$. Then by Power of a Point, $PA^2 = (PD)(PE)$.

Next, we have that $\angle BDC = 90^\circ = \angle BEC$, so $BCDE$ is cyclic. Since M is the midpoint of BC, the circumcenter of the $BCDE$ is M. Then if r is the radius of that circumcircle of $BCDE$, by Power of a Point, $(PD)(PE) = (PM - r)(PM + r) = PM^2 - r^2$. Since $r = \frac{BC}{2} = 24$, we can combine our results to get

$$PA^2 = PM^2 - r^2 \implies PM = \sqrt{10^2 + 24^2} = \boxed{26}$$