1. Pentagon $ABCDE$ has $AB = BC = CD = DE$, $\angle ABC = \angle BCD = 108°$, and $\angle CDE = 168°$. Find the measure of angle $\angle BEA$ in degrees.

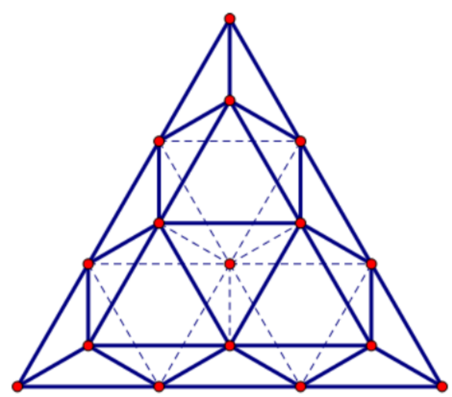
   Answer: 24

   Solution: Construct point F such that $\angle CDF = 108°$, $\angle FDE = 60°$, and $DF = DE$. Then triangle FED is equilateral. Note that BE is a line of symmetry for hexagon $ABCDEF$, so $\angle BEF = 30°$. Furthermore, since $\triangle AFE$ is isosceles we see that $\angle AEF = 6°$. Thus $\angle BEA = \angle BEF - \angle AEF = 24°$.

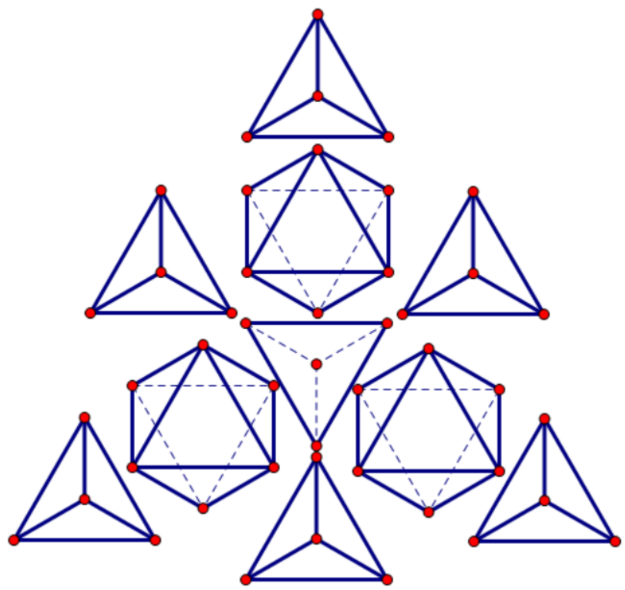
2. On each edge of a regular tetrahedron, five points that separate the edge into six equal segments are marked. There are twenty planes that are parallel to a face of the tetrahedron and pass through exactly three of the marked points. When the tetrahedron is cut along each of these twenty planes, how many new tetrahedrons are produced?

   Answer: 76

   Solution: Consider the five cuts in the same direction, which cut the tetrahedron into five frustums of different size and a unit regular tetrahedron. Note that the i-th smallest frustum has a smaller base being an equilateral triangle with side length i and a larger base being an equilateral triangle with side length $i+1$. Now perform the cuts in the remaining directions on each frustum. Below is an example for the cuts on second smallest frustum.



   After that, on the smaller base there are $\frac{i(i-1)}{2}$ downward pointing triangles and each of them faces an intersection on the larger base, meaning that each correspond to a tetrahedron; on the larger base there are $\frac{(i+1)(i+2)}{2}$ upward-pointing triangles, each facing an intersection and therefore corresponding to a tetrahedron as well. The $\frac{i(i+1)}{2}$ upward-pointing triangles on the smaller base and downward-pointing triangles on the larger base correspond to each other and each pair correspond to a regular octahedron. For example, when we separate all the pieces in the second frustum, we should get the following pieces: (in their relative position in order to aid visualization)



Finally, by summing $\frac{i(i-1)}{2}$ and $\frac{(i+1)(i+2)}{2}$ over $i = 0, 1, 2, 3, 4, 5$, we get that there are $(0 + 0 + 1 + 3 + 6 + 10) + (1 + 3 + 6 + 10 + 15 + 21) = 76$ tetrahedrons (and $0 + 1 + 3 + 6 + 10 + 15 = 35$ octahedrons).

3. Three cities that are located on the vertices of an equilateral triangle with side length 100 units. A missile flies in a straight line in the same plane as the equilateral triangle formed by the three citiies. The radar from City A reported that the closest approach of the missile was 20 units. The radar from City B reported that the closest approach of the missile was 60 units. However, the radar for city C malfunctioned and did not report a distance. Find the minimum possible distance for the closest approach of the missile to city C.

Answer: $\mathbf{30\sqrt{3} - 20}$

Solution: We first begin by drawing a circle of radius 20 around city A and a circle of radius 60 around city B. The path of the missile must be one of the four (interior or exterior) tangents to these two circles. We will focus on calculating the distance from C to the nearest tangent, since the distance from C to the other tangents can be calculated similarly.

Let the tangent intersect segment AB at D, and let the tangent point of the circle centered at A touch the line at A' and the tangent point of the circle centered at B touch the line at B'. Then, triangle $BB'D$ is similar to triangle $AA'D$ with ratio $1 : 3$. Therefore, $AD = 25$ and $DB = 75$ and we see that the triangles are in fact $3 - 4 - 5$ right triangles.

Now, let the path of the missile interesect segment BC at E, and draw a circle around C such that the circle is also tangent to the path of the missile. Once again, denote the point of tangency as C'. Then, triangle $CC'E$ is similar to $BB'E$. Let $CC' = r$ be the radius of the circle around C (which is also the answer we want).

We can then write the equation $\frac{r}{\cos(60^\circ - \theta)} + \frac{60}{\cos(60^\circ - \theta)} = 100$ where $\theta = \angle B'BD$. By using angle sum formulas, this reduces down to $r + 60 = 100\cos(60^\circ - \theta) = 40 + 30\sqrt{3}$. Our desired distance is thus $r = 30\sqrt{3} - 20$.