1. If $f(x) = (x-1)^4(x-2)^3(x-3)^2$, find $f'''(1) + f''(2) + f'(3)$.

   Answer: 0

   A polynomial $p(x)$ has a multiple root at $x = a$ if and only if $x - a$ divides both p and p'. Continuing inductively, the nth derivative $p^{(n)}$ has a multiple root b if and only if $x - b$ divides $p^{(n)}$ and $p^{(n+1)}$. Since $f(x)$ has 1 as a root with multiplicity 4, $x-1$ must divide each of f, f', f'', f'''. Hence $f'''(1) = 0$. Similarly, $x-2$ divides each of f, f', f'' so $f''(2) = 0$ and $x-3$ divides each of f, f', meaning $f'(3) = 0$. Hence the desired sum is 0.

2. A trapezoid is inscribed in a semicircle of radius 2 such that one base of the trapezoid lies along the diameter of the semicircle. Find the largest possible area of the trapezoid.

   Answer: $3\sqrt{3}$

   Clearly, a trapezoid with maximal area will have a base equal to the diameter. If x is the height of the trapezoid, then the area of a trapezoid is $\frac{h(b_1+b_2)}{2} = A(x) = \left(2 + \sqrt{4-x^2}\right) \cdot x$ so the maximum occurs when

   $$0 = A'(x) = 2 + \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} = \frac{2\sqrt{4-x^2} + 4 - 2x^2}{\sqrt{4-x^2}},$$

   which is equivalent to

   $$4(4-x^2) = (2x^2-4)^2 = 4x^4 - 16x^2 + 16.$$

   Collecting like terms gives $4x^4 = 12x^2$, and since $x \neq 0$ (the degenerate case), we get that $x = \sqrt{3}$. Thus the desired maximum occurs at $x = \sqrt{3}$ and so the maximum area is

   $$A(\sqrt{3}) = \left(2 + \sqrt{4-3}\right) \cdot \sqrt{3} = 3\sqrt{3}.$$

3. A sector of a circle has angle θ. Find the value of θ, in radians, for which the ratio of the sector's area to the square of its perimeter (the arc along the circle and the two radial edges) is maximized. Express your answer as a number between 0 and 2π.

   Answer: 2

   Suppose that the circle has radius r. Then the area of the circle is πr^2, so the area of the sector is $\frac{\theta}{2\pi}\pi r^2 = \frac{1}{2}\theta r^2$. The arc of the perimeter of the sector has length $\frac{\theta}{2\pi}2\pi r = \theta r$, and the two straight edges of the sector each has length r, so the perimeter has length $\theta r + 2r = (\theta+2)r$, and hence the square of the perimeter is $(\theta+2)^2 r^2$. The ratio that we want to maximize is therefore

   $$\frac{\frac{1}{2}\theta r^2}{(\theta+2)^2 r^2} = \frac{\theta}{2(\theta+2)^2}.$$

   To do this, differentiate to find the critical points:

   $$0 = \frac{d}{d\theta}\left(\frac{\theta}{2(\theta+2)^2}\right) = \frac{2(\theta+2)^2 - 4\theta(\theta+2)}{4(\theta+2)^4} = \frac{2(\theta+2) - 4\theta}{4(\theta+2)^3} = \frac{2-\theta}{2(\theta+2)^3} \implies \theta = 2.$$

   Observe that the derivative is decreasing at $\theta = 2$, which implies that this is a local maximum, as desired.

   Alternate Solution:

   Equivalently, we can minimize the reciprocal:

   $$0 = \frac{d}{d\theta}\left(\frac{2(\theta+2)^2}{\theta}\right) = 2\frac{d}{d\theta}\left(4\theta^{-1} + 4 + \theta\right) = 2\left(-4\theta^{-2} + 1\right) \implies \theta^2 = 4 \implies \theta = 2.$$

4. Let $f(x) = \frac{x^3 e^{x^2}}{1-x^2}$. Find $f^{(7)}(0)$, the 7th derivative of f evaluated at 0.

   Answer: 12600

Since $f^{(n)}(0) = a_n n!$, where a_n is the nth Taylor series coefficient, we just need to find the Taylor series of f and read off the appropriate coefficient. The Taylor series is given by

$$f(x) = x^3\left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \cdots\right)\left(1 + x^2 + x^4 + \cdots\right).$$

The coefficient of x^7 is $\frac{1}{2!} + \frac{1}{1!} + 1 = \frac{5}{2}$, so $f^{(7)}(0) = 7! \cdot \frac{5}{2} = 12600$.

5. The real-valued infinitely differentiable function $f(x)$ is such that $f(0) = 1$, $f'(0) = 2$, and $f''(0) = 3$. Furthermore, f has the property that

$$f^{(n)}(x) + f^{(n+1)}(x) + f^{(n+2)}(x) + f^{(n+3)}(x) = 0$$

for all $n \geq 0$, where $f^{(n)}(x)$ denotes the nth derivative of f. Find $f(x)$.

Answer: $\boldsymbol{2e^{-x} - \cos x + 4\sin x}$

We solve the differential equation $f + f' + f'' + f''' = 0$. Let $f + f' = g$. Then we need to solve $g + g'' = 0$, which has solution $g(x) = a\cos x + b\sin x$. Then

$$e^x(f + f') = (e^x f)' = ae^x\cos x + be^x\sin x,$$

so that

$$f = e^{-x}\left(\int (ae^x\cos x + be^x\sin x)\,dx + c\right) = ce^{-x} + a'\cos x + b'\sin x.$$

Finally, we find $f(0) = c + a'$, $f'(0) = -c + b'$, and $f''(0) = c - a'$ and solve for a', b', c.

Alternate Solution: Observe that since the given equation holds for all n, by moving the index up one and then subtracting, we get $f^{(n)}(x) - f^{(n+4)}(x) = 0$, so that $f^{(n)}(x) = f^{(n+4)}(x)$. That is, any function that satisfies the given equation must also have the property that the derivatives repeat in cycles of 4. However, as we will see, this is only a necessary property, not a sufficient one. The characteristic equation of the given differential equation is $\lambda^{n+4} - \lambda^n = 0$, or $\lambda^n(\lambda^4 - 1) = 0$. The roots of this equation are 0 and the fourth roots of unity, so a complete set of solutions is given by $f(x) = ae^x + be^{-x} + ce^{ix} + de^{-ix}$ (the terms e^{ix} and e^{-ix} can be written in terms of sine and cosine, as is boxed above). Note however, that ae^x does not satisfy the original differential equation as all of its derivatives have the same sign. Relabelling the constants, the solution set is $f(x) = ae^{-x} + be^{ix} + ce^{-ix} = ae^{-x} + b(\cos x + i\sin x) + c(\cos x - i\sin x)$.

6. Compute $\displaystyle\int_{-\pi}^{\pi} \frac{x^2}{1 + \sin x + \sqrt{1 + \sin^2 x}}\,dx$.

Answer: $\boldsymbol{\frac{\pi^3}{3}}$

Use symmetry around the origin. Substitute x to $-x$, so the integral is now

$$\int_{-\pi}^{\pi} \frac{x^2\,dx}{1 - \sin x + \sqrt{1 + \sin^2 x}}.$$

Add the two integrals, and note that

$$\frac{1}{1 + \sin x + \sqrt{1 + \sin^2 x}} + \frac{1}{1 - \sin x + \sqrt{1 + \sin^2 x}} = \frac{2 + 2\sqrt{1 + \sin^2 x}}{2 + \sin^2 x + 2\sqrt{1 + \sin^2 x} - \sin^2 x} = 1,$$

so the integral is the same as $\frac{1}{2}\int_{-\pi}^{\pi} x^2 dx = \frac{\pi^3}{3}$.

7. For the curve $\sin(x) + \sin(y) = 1$ lying in the first quadrant, find the constant α such that

$$\lim_{x \to 0} x^\alpha \frac{d^2y}{dx^2}$$

exists and is nonzero.

Answer: $\boldsymbol{\frac{3}{2}}$

Differentiate the equation to get

$$\cos(x) + \frac{dy}{dx}\cos(y) = 0$$

and again to get

$$-\sin(x) + \frac{d^2y}{dx^2}\cos(y) - \left(\frac{dy}{dx}\right)^2 \sin(y) = 0.$$

By solving these we have

$$\frac{dy}{dx} = -\frac{\cos(x)}{\cos(y)}$$

and

$$\frac{d^2y}{dx^2} = \frac{\sin(x)\cos^2(y) + \sin(y)\cos^2(x)}{\cos^3(y)}.$$

Let $\sin(x) = t$, then $\sin(y) = 1 - t$. Also $\cos(x) = \sqrt{1-t^2}$ and $\cos(y) = \sqrt{1-(1-t)^2} = \sqrt{t(2-t)}$. Substituting gives

$$\frac{d^2y}{dx^2} = \frac{t^2(2-t) + (1-t)(1-t^2)}{t^{3/2}(2-t)^{3/2}} = t^{-3/2}\frac{1-t+t^2}{(2-t)^{3/2}}.$$

Since $\lim_{x\to 0} \frac{t}{x} = 1$, $\alpha = \frac{3}{2}$ should give the limit $\lim_{x\to 0} x^\alpha \frac{d^2y}{dx^2} = \frac{1}{2\sqrt{2}}$.

8. Compute $\displaystyle\int_{\frac{1}{2}}^{2} \frac{\tan^{-1} x}{x^2 - x + 1}\, dx$.

Answer: $\boldsymbol{\frac{\pi^2\sqrt{3}}{18}}$

Take $y = 1/x$, then $\frac{dx}{x^2-x+1} = -\frac{dy}{y^2-y+1}$. Note furthermore by the tangent addition formula that $\tan^{-1}(x) + \tan^{-1}(y) = \pi/2$. The original integral is equal to the average of these two integrals:

$$\frac{1}{2}\left(\int_{\frac{1}{2}}^{2} \frac{\tan^{-1} x}{x^2 - x + 1}\, dx + \int_{\frac{1}{2}}^{2} \frac{\frac{\pi}{2} - \tan^{-1} y}{y^2 - y + 1}\, dy\right) = \frac{\pi}{4}\int_{1/2}^{2} \frac{dx}{x^2 - x + 1}.$$

Substitute $x = \frac{\sqrt{3}}{2}\theta + 1/2$, then

$$\frac{\pi}{4}\int_{1/2}^{2} \frac{dx}{x^2 - x + 1} = \frac{\pi}{4}\frac{4}{3}\frac{\sqrt{3}}{2}\int_{0}^{\sqrt{3}} \frac{1}{\theta^2 + 1}\, d\theta = \frac{\pi^2\sqrt{3}}{18}.$$

9. Solve the integral equation

$$f(x) = \int_0^x e^{x-y} f'(y)\, dy - (x^2 - x + 1)e^x.$$

Answer: $\boldsymbol{f(x) = (2x-1)e^x}$

Differentiate both sides to get

$$f'(x) = \frac{d}{dx} e^x \int_0^x e^{-y} f'(y)\, dy - \frac{d}{dx}(x^2 - x + 1)e^x$$

$$f'(x) = f'(x) + \int_0^x e^{x-y} f'(y)\, dy - (x^2 + x)e^x.$$

But

$$\int_0^x e^{x-y} f'(y)\, dy = f(x) + (x^2 - x + 1)e^x$$

so by substituting it we get

$$f(x) + (x^2 - x + 1)e^x - (x^2 + x)e^x = 0,$$

and $f(x) = (2x - 1)e^x$.

10. Compute the integral

$$\int_0^{\pi} \ln(1 - 2a\cos x + a^2)\,dx$$

for $a > 1$.

Answer: $\mathbf{2\pi \ln a}$

Solution 1:

This integral can be computed using a Riemann sum. Divide the interval of integration $[0, \pi]$ into n parts to get the Riemann sum

$$\frac{\pi}{n}\left[\ln\left(a^2 - 2a\cos\frac{\pi}{n} + 1\right) + \ln\left(a^2 - 2a\cos\frac{2\pi}{n} + 1\right) + \cdots + \ln\left(a^2 - 2a\cos\frac{(n-1)\pi}{n} + 1\right)\right].$$

Recall that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

We can rewrite this sum of logs as a product and factor the inside to get

$$\frac{\pi}{n}\ln\left[\prod_{k=1}^{n-1}\left(a^2 - 2a\cos\frac{k\pi}{n} + 1\right)\right] = \frac{\pi}{n}\ln\left[\prod_{k=1}^{n-1}\left(a - e^{k\pi i/n}\right)\left(a - e^{-k\pi i/n}\right)\right].$$

The terms $e^{\pm k\pi i/n}$ are all of the $2n$-th roots of unity except for ± 1, so the inside product contains all of the factors of $a^{2n} - 1$ except for $a - 1$ and $a + 1$. The Riemann sum is therefore equal to

$$\frac{\pi}{n}\ln\frac{a^{2n} - 1}{a^2 - 1}$$

To compute the value of the desired integral, we compute the limit of the Riemann sum as $n \to \infty$; this is

$$\lim_{n\to\infty}\frac{\pi}{n}\ln\frac{a^{2n} - 1}{a^2 - 1} = \lim_{n\to\infty}\pi\ln\sqrt[n]{\frac{a^{2n} - 1}{a^2 - 1}} = \lim_{n\to\infty}\pi\ln a^2 = 2\pi\ln a.$$

(This is problem 471 of Răzvan Gelca and Titu Andreescu's book *Putnam and Beyond*. The solution is due to Siméon Poisson.)

Solution 2:

Let the desired integral be $I(a)$, where we think of this integral as a function of the parameter a. In this solution, we differentiate by a to convert the desired integral to an integral of a rational function in $\cos x$:

$$\frac{d}{da}I(a) = \frac{d}{da}\int_0^{\pi}\ln(1 - 2a\cos x + a^2)\,dx = \int_0^{\pi}\frac{2a - 2\cos x}{1 - 2a\cos x + a^2}\,dx.$$

All integrals of this form can be computed using the substitution $t = \tan\frac{x}{2}$. Then $x = 2\arctan t$, so $dx = \frac{2}{1+t^2}\,dt$ and

$$\cos x = \cos(2\arctan t) = 2\cos(\arctan t)^2 - 1 = 2\left(\frac{1}{1+t^2}\right) - 1 = \frac{1-t^2}{1+t^2},$$

so our integral becomes

$$\frac{d}{da}I(a) = \int_0^{\infty}\frac{2a - 2\frac{1-t^2}{1+t^2}}{1 - 2a\frac{1-t^2}{1+t^2} + a^2}\frac{2}{1+t^2}\,dt = 4\int_0^{\infty}\frac{a(1+t^2) - (1-t^2)}{(1+t^2) - 2a(1-t^2) + a^2(1+t^2)}\frac{1}{1+t^2}\,dt$$

$$= 4\int_0^{\infty}\frac{(a+1)t^2 + (a-1)}{((a+1)^2t^2 + (a-1)^2)(1+t^2)}\,dt = \frac{2}{a}\int_0^{\infty}\frac{a^2-1}{(a+1)^2t^2 + (a-1)^2}\,dt + \frac{2}{a}\int_0^{\infty}\frac{1}{1+t^2}\,dt.$$

In the first integral, we do the substitution $t = \frac{a-1}{a+1}u$. Then $dt = \frac{a-1}{a+1}du$ and we have

$$= \frac{2}{a}\int_0^\infty \frac{1}{1+u^2}\,du + \frac{2}{a}\int_0^\infty \frac{1}{1+t^2}\,dt = \frac{2}{a}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{2\pi}{a}.$$

Therefore, our desired integral is the integral of the previous quantity, or

$$I = \int_0^\pi \ln(1-2a\cos x + a^2)\,dx = 2\pi \ln a.$$

Solution 3:

We use Chebyshev polynomials[1]. First, define the Chebyshev polynomial of the first kind to be $T_n(x) = \cos(n \arccos x)$. This is a polynomial in x, and note that $T_n(\cos x) = \cos(nx)$. Note that

$$\cos((n+1)x) = \cos nx \cos x - \sin nx \sin x$$
$$\cos((n-1)x) = \cos nx \cos x + \sin nx \sin x,$$

so that $\cos((n+1)x) = 2\cos nx \cos x - \cos((n-1)x)$ and hence the Chebyshev polynomials satisfy the recurrence $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

Therefore, the Chebyshev polynomials satisfy the generating function

$$\sum_{n=0}^\infty T_n(x)t^n = \frac{1-tx}{1-2tx+t^2}.$$

Now, substituting $x \mapsto \cos x$ and $t \mapsto a^{-1}$, we have

$$\sum_{n=0}^\infty \cos(nx)a^{-n} = a\frac{a-\cos x}{a^2-2a\cos x+1}.$$

So

$$2\sum_{n=0}^\infty \cos(nx)a^{-n-1} = \frac{2a-2\cos x}{1-2a\cos x+a^2}.$$

Then

$$\int_0^\pi \frac{2a-2\cos x}{1-2a\cos x+a^2}\,dx = 2\int_0^\pi \sum_{n=0}^\infty \cos(nx)a^{-n-1}\,dx = 2\sum_{n=0}^\infty \left(a^{-n-1}\int_0^\pi \cos(nx)\,dx\right) = 2\pi a^{-1}.$$

Now, since

$$\ln(1-2a\cos x+a^2) = \int \frac{2a-2\cos x}{1-2a\cos x+a^2}\,da,$$

we see that

$$\int_0^\pi \ln(1-2a\cos x+a^2)\,dx = \int 2\pi a^{-1}\,da = 2\pi \ln a.$$

Solution 4:

We can also give a solution based on physics. By symmetry, we can evaluate the integral from 0 to 2π and divide the answer by 2, so

$$\int_0^\pi \ln(1-2a\cos x+a^2)\,dx = \int_0^{2\pi} \ln\sqrt{1-2a\cos x+a^2}\,dx.$$

Now let's calculate the 2D gravitational potential of a point mass falling along the x axis towards a unit circle mass centered around the origin. We set the potential at infinity to 0. We also note that,

[1] http://en.wikipedia.org/wiki/Chebyshev_polynomials

since the 2D gravitational force between two masses is proportional to $\frac{1}{r}$, the potential between two masses is proportional to $-\ln r$. So to calculate the gravitational potential, we integrate $-\ln r$ over the unit circle. But if the point mass is at $(a, 0)$, then the distance between the point mass and the section of the circle at angle x is $\sqrt{1-2a\cos x+a^2}$. So we get the integral

$$-\int_0^{2\pi} \ln\sqrt{1-2a\cos x+a^2}\,dx$$

This is exactly the integral we want to calculate! We can also calculate this potential by concentrating the mass of the circle at its center. The circle has mass 2π and its center is distance a from the point mass. So the potential is simply $-2\pi\ln a$. Thus, the final answer is $2\pi\ln(a)$.

Solution 5:

This problem also has a solution which uses the Residue Theorem from complex analysis. It is easy to show that

$$2\int_0^{\pi} \ln(1-2a\cos(x)+a^2)\,dx = \int_0^{2\pi} \ln(1-2a\cos(x)+a^2)\,dx.$$

Furthermore, observe that $1-2a\cos x+a^2 = (a-e^{ix})(a-e^{-ix})$. Thus, our integral is

$$I = \frac{1}{2}\left(\int_0^{2\pi} \ln[(a-e^{ix})(a-e^{-ix})]dx\right) = \frac{1}{2}\left(\int_0^{2\pi} \ln(a-e^{ix})dx + \int_0^{2\pi} \ln(a-e^{-ix})dx\right),$$

where the integrals are performed on the real parts of the logarithms in the second expression. In the first integral, substitute $z = e^{ix}$, $dz = ie^{ix}\,dx = iz\,dx$; the resulting contour integral is

$$\oint_{\|z\|=1} \frac{\ln(a-z)}{iz}dz.$$

By the Residue Theorem, this is equal to $2\pi i\,\mathrm{Res}_{z=0}\,\frac{\ln(a-z)}{iz} = 2\pi\ln(a)$. The second integral is identical. Thus, the final answer is $\frac{1}{2}(4\pi\ln(a)) = 2\pi\ln(a)$.