**Time limit:** 110 minutes.
**Instructions:** This test contains 25 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
**No calculators.**

1. Let $F(x)$ be a real-valued function defined for all real $x \neq 0, 1$ such that
$$F(x) + F\left(\frac{x-1}{x}\right) = 1 + x.$$
Find $F(2)$.

2. Given that $a_1 = 2$, $a_2 = 3$, $a_n = a_{n-1} + 2a_{n-2}$, what is $a_{100} + a_{99}$?

3. Let sequence $A$ be $\{\frac{7}{4}, \frac{7}{6}, \frac{7}{9}, \ldots\}$ where the $j^{th}$ term is given by $a_j = \frac{7}{4}\left(\frac{2}{3}\right)^{j-1}$. Let $B$ be a sequence where the $j^{th}$ term is defined by $b_j = a_j^2 + a_j$. What is the sum of all the terms in $B$?

4. Find all rational roots of $|x-1| \times |x^2 - 2| - 2 = 0$.

5. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. In how many ways can two (not necessarily distinct) elements $a, b$ be taken from $S$ such that $\frac{a}{b}$ is in lowest terms, i.e. $a$ and $b$ share no common divisors other than 1?

6. Find all square numbers which can be represented in the form $2^a + 3^b$, where $a, b$ are nonnegative integers. You can assume the fact that the equation $3^x - 2^y = 1$ has no integer solutions if $x \geq 3$.

7. A frog is jumping on the number line, starting at zero and jumping to seven. He can jump from $x$ to either $x+1$ or $x+2$. However, the frog is easily confused, and before arriving at the number seven, he will turn around and jump in the wrong direction, jumping from $x$ to $x-1$. This happens exactly once, and will happen in such a way that the frog will not land on a negative number. How many ways can the frog get to the number seven?

8. Call a nonnegative integer $k$ *sparse* when all pairs of 1's in the binary representation of $k$ are separated by at least two zeroes. For example, $9 = 1001_2$ is sparse, but $10 = 1010_2$ is not sparse. How many sparse numbers are less than $2^{17}$?

9. Two ants begin on opposite corners of a cube. On each move, they can travel along an edge to an adjacent vertex. Find the probability they both return to their starting position after 4 moves.

10. An unfair coin has a 2/3 probability of landing on heads. If the coin is flipped 50 times, what is the probability that the total number of heads is even?

11. Find the unique polynomial $P(x)$ with coefficients taken from the set $\{-1, 0, 1\}$ and with least possible degree such that $P(2010) \equiv 1 \pmod 3$, $P(2011) \equiv 0 \pmod 3$, and $P(2012) \equiv 0 \pmod 3$.

12. Let $a, b \in \mathbb{C}$ such that $a + b = a^2 + b^2 = \frac{2\sqrt{3}}{3}i$. Compute $|\operatorname{Re}(a)|$.

13. Let $T_n$ denote the number of terms in $(x+y+z)^n$ when simplified, i.e. expanded and like terms collected, for non-negative integers $n \geq 0$. Find
$$\sum_{k=0}^{2010} (-1)^k T_k$$
$$= T_0 - T_1 + T_2 - \cdots - T_{2009} + T_{2010}.$$

14. Let $M = (-1, 2)$ and $N = (1, 4)$ be two points in the plane, and let $P$ be a point moving along the $x$-axis. When $\angle MPN$ takes on its maximum value, what is the $x$-coordinate of $P$?

15. Consider the curves $x^2 + y^2 = 1$ and $2x^2 + 2xy + y^2 - 2x - 2y = 0$. These curves intersect at two points, one of which is $(1, 0)$. Find the other one.

16. If $r$, $s$, $t$, and $u$ denote the roots of the polynomial $f(x) = x^4 + 3x^3 + 3x + 2$, find
$$\frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2} + \frac{1}{u^2}.$$

17. An *icosahedron* is a regular polyhedron with 12 vertices, 20 faces, and 30 edges. How many rigid rotations $G$ are there for an icosahedron in $\mathbb{R}^3$?

18. Pentagon $ABCDE$ is inscribed in a circle of radius 1. If $\angle DEA \cong \angle EAB \cong \angle ABC$, $m\angle CAD = 60°$, and $BC = 2DE$, compute the area of $ABCDE$.

19. Five students at a meeting remove their name tags and put them in a hat; the five students then each randomly choose one of the name tags from the bag. What is the probability that exactly one person gets their own name tag?

20. Find the 2011th-smallest $x$, with $x > 1$, that satisfies the following relation:
$$\sin(\ln x) + 2\cos(3\ln x)\sin(2\ln x) = 0.$$

21. An ant is leashed up to the corner of a solid square brick with side length 1 unit. The length of the ant's leash is 6 units, and it can only travel on the ground and not through or on the brick. In terms of $x = \arctan\left(\frac{3}{4}\right)$, what is the area of region accessible to the ant?

22. Compute the sum of all $n$ for which the equation $2x + 3y = n$ has exactly 2011 nonnegative $(x, y \geq 0)$ integer solutions.

23. Let $ABC$ be any triangle, and $D, E, F$ be points on $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $CD = 2BD$, $AE = 2CE$ and $BF = 2AF$. $\overline{AD}$ and $\overline{BE}$ intersect at $X$, $\overline{BE}$ and $\overline{CF}$ intersect at $Y$, and $\overline{CF}$ and $\overline{AD}$ intersect at $Z$. Find $\frac{Area(\triangle ABC)}{Area(\triangle XYZ)}$.

24. Let $P(x)$ be a polynomial of degree 2011 such that $P(1) = 0$, $P(2) = 1$, $P(4) = 2$, ... , and $P(2^{2011}) = 2011$. Compute the coefficient of the $x^1$ term in $P(x)$.

25. Find the maximum of
$$\frac{ab + bc + cd}{a^2 + b^2 + c^2 + d^2}$$
for reals $a$, $b$, $c$, and $d$ not all zero.