1. What is $\int_0^{10}(x-5)+(x-5)^2+(x-5)^3\,dx$?

   Answer: $\frac{250}{3}$

   Solution: This integral is equal to

$$\int_{-5}^{5} x+x^2+x^3\,dx = \int_{-5}^{5} x^2\,dx = \left(\frac{5^3}{3}-\frac{(-5)^3}{3}\right) = \boxed{\frac{250}{3}}.$$

2. Find the maximum value of

$$\int_{-\pi/2}^{3\pi/2} \sin(x)f(x)\,dx$$

   subject to the constraint $|f(x)| \leq 5$.

   Answer: 20

   Solution: Clearly we want to maximize $f(x)$ when $\sin(x) \geq 0$ and minimize $f(x)$ when $\sin(x) < 0$. We do this by setting $f(x) = 5$ in the first case and $f(x) = -5$ in the second case. Noting that the bounds of integration cover precisely one full period of sin, we see that the integral becomes equivalent to twice the integral of $5\sin(x)$ over the half period where $\sin(x) \geq 0$. This results in $\boxed{20}$.

3. Calculate

$$\int_{2^5}^{3^5} \frac{1}{x-x^{3/5}}\,dx.$$

   Answer: $\frac{5}{2}\ln\frac{8}{3}$

   Solution: Note that we can write the integral as

$$\int_{2^5}^{3^5} \frac{1}{x^{3/5}(x^{2/5}-1)}\,dx.$$

   We solve via u-substitution. Let $u = x^{2/5}-1$:

$$du = \frac{2}{5}x^{-3/5}\,dx \implies dx = \frac{5}{2}x^{3/5}\,du.$$

   The integral becomes

$$\frac{5}{2}\int_{2^2-1}^{3^2-1} \frac{x^{3/5}}{x^{3/5}\cdot u}\,du = \frac{5}{2}\int_3^8 \frac{1}{u}\,du,$$

   which evaluates to

$$\frac{5}{2}(\ln 8 - \ln 3) = \boxed{\frac{5}{2}\ln\frac{8}{3}}.$$

4. Compute the x-coordinate of the point on the curve $y = \sqrt{x}$ that is closest to the point $(2,1)$.

   Answer: $\frac{2+\sqrt{3}}{2}$

   Solution: We want to minimize the distance between the points (a^2, a) and $(2,1)$. We can equivalently minimize the square of the distance between those two points, which is

$$(2-a^2)^2 + (1-a)^2 = a^4 - 3a^2 - 2a + 5.$$

The derivative of this function is $4a^3-6a-2$, which can be factored as $2(a+1)(2a^2-2a-1)$. The roots of this cubic are therefore $a=-1, \dfrac{1\pm\sqrt{3}}{2}$. Two of the roots are negative and therefore invalid, so $a=\dfrac{1+\sqrt{3}}{2}$ and $a^2=\boxed{\dfrac{2+\sqrt{3}}{2}}$.

5. Let

$$f(x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5},$$

and set $g(x)=f^{-1}(x)$. Compute $g^{(3)}(0)$.

Answer: 1

Solution 1: The inverse function rule tells us that

$$g'(x)=\left[f'(g(x))\right]^{-1}.$$

Using this and the fact that $g(0)$ is clearly equal to zero, this problem can be solved with a straightforward bash.

Solution 2: We begin by observing that we know by the inverse function rule that we will not need to know any derivatives of f past its third derivative. Since the first three derivatives of f at zero agree with those of the function $-\log(1-x)$ (by Taylor series expansion), we can assume $f(x)=-\log(1-x)$. Now,

$$y=-\log(1-x) \implies -y=\log(1-x) \implies e^{-y}=1-x \implies x=1-e^{-y}$$

so $g(y)=1-e^{-y}$ and therefore $g^{(3)}(0)=e^{-0}=\boxed{1}$.

6. Compute

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}.$$

Answer: $e^{-1/3}$

Solution 1: We take logs and evaluate by L'Hopital's rule:

$$\lim_{x\to 0}\log\left[\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}\right]=\lim_{x\to 0}\frac{\log(\sin x)-\log x}{1-\cos x}=\lim_{x\to 0}\frac{\frac{\cos x}{\sin x}-\frac{1}{x}}{\sin x}=\lim_{x\to 0}\frac{x\cos x-\sin x}{x\sin^2 x}$$
$$=\lim_{x\to 0}\frac{-x\sin x}{\sin^2 x+2x\sin x\cos x}=\lim_{x\to 0}\frac{-x}{\sin x+2x\cos x}=\lim_{x\to 0}\frac{-1}{\cos x+2\cos x-2x\sin x}=-\frac{1}{3}.$$

Therefore, the answer is $e^{-1/3}$.

Solution 2: We can approximate $\sin x$ and $\cos x$ by their Taylor series. Applying substitutions then yields

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}=\lim_{x\to 0}\left(1-\frac{x^2}{6}\right)^{\frac{2}{x^2}}=\lim_{x\to 0}\left(1+\frac{x^2}{6}\right)^{-\frac{2}{x^2}}$$
$$=\lim_{x\to\infty}\left(1+\frac{1}{6x^2}\right)^{-2x^2}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{-x/3}=\boxed{e^{-1/3}}$$

because $e=\lim_{x\to\infty}(1+\frac{1}{x})^x$.

7. A differentiable function g satisfies

$$\int_0^x (x-t+1)g(t)\,dt = x^4 + x^2$$

for all $x \geq 0$. Find $g(x)$.

Answer: $\mathbf{12x^2 - 24x + 26 - 26e^{-x}}$

Solution: First differentiate the equation with respect to x:

$$g(x) + \int_0^x g(t)\,dt = 4x^3 + 2x.$$

Differentiate again to obtain

$$g'(x) + g(x) = 12x^2 + 2.$$

The solution $12x^2 - 24x + 26$ can be found using the method of undetermined coefficients, so the general solution will be

$$g(x) = 12x^2 - 24x + 26 + Ce^{-x}$$

for some constant C. By substituting $x = 0$ into the first equation, we see that $g(0) = 0$. We therefore find that $C = -26$, making the answer $\boxed{12x^2 - 24x + 26 - 26e^{-x}}$.

8. Compute

$$\int_0^\infty \frac{\ln x}{x^2+4}\,dx.$$

Answer: $\mathbf{\frac{\pi \ln 2}{4}}$

Solution: Substitute $x = 2\tan\theta$ to get

$$\int_0^\infty \frac{\ln x}{x^2+4}\,dx = \frac{1}{2}\int_0^{\pi/2} \ln(2\tan\theta)\,d\theta = \frac{1}{2}\cdot\frac{\pi}{2}\ln 2 + \frac{1}{2}\int_0^{\pi/2}\ln(\tan\theta)\,d\theta.$$

We will now show that this final integral is zero by substituting $u = \pi/2 - \theta$ to yield

$$\begin{aligned}\int_0^{\pi/2}\ln(\tan\theta)\,d\theta &= \int_0^{\pi/2}\ln\left(\tan\left(\frac{\pi}{2}-\theta\right)\right)\,d\theta \\ &= \int_0^{\pi/2}\ln\left(\frac{1}{\tan\theta}\right)\,d\theta = -\int_0^{\pi/2}\ln(\tan\theta)\,d\theta,\end{aligned}$$

which gives us what we wanted, so the answer is therefore $\boxed{\dfrac{\pi\ln 2}{4}}$.

9. Find the ordered pair (α, β) with non-infinite $\beta \neq 0$ such that $\displaystyle\lim_{n\to\infty}\frac{\sqrt[n^2]{1!2!\cdots n!}}{n^\alpha} = \beta$ holds.

Answer: $\mathbf{(1/2, e^{-3/4})}$

Solution 1: Taking the logarithm of $1!2!\cdots n!$, we find

$$\begin{aligned}\ln(1!2!\cdots n!) &= \ln 1 + (\ln 1 + \ln 2) + (\ln 1 + \ln 2 + \ln 3) + \cdots \\ &= n\ln 1 + (n-1)\ln 2 + \cdots + \ln n \\ &= n\ln\frac{1}{n} + (n-1)\ln\frac{2}{n} + \cdots + \ln\frac{n}{n} + \frac{n(n+1)}{2}\ln n.\end{aligned}$$

Dividing this by n^2, we have

$$\ln(\sqrt[n^2]{1!2!\cdots n!}) = \frac{n+1}{2n}\ln n + \frac{1}{n}\left(\sum_{m=1}^{n}\frac{n+1-m}{n}\ln\frac{m}{n}\right)$$

As n goes to infinity, the sum will converge to the integral

$$\int_0^1 (1-x)\ln x\,dx = \left[\frac{1}{4}x^2 - \frac{1}{2}x^2\ln x - x + x\ln x\right]_0^1 = -\frac{3}{4}$$

and the first term will approach $\frac{1}{2}\ln n$, so if we subtract $\frac{1}{2}\ln n$ then this expression will converge to $-\frac{3}{4}$. Finally, by raising e to the power of each side, we have

$$\lim_{n\to\infty}\frac{\sqrt[n^2]{1!2!\cdots n!}}{n^{1/2}} = \boxed{e^{-3/4}}.$$

Solution 2: We take logs and approximate by Stirling's approximation[1]. Here, we will be somewhat sloppy with our approximations, and freely use $\sim$ to indicate approximations that become irrelevant in the limit. All of these calculations can be made rigorous, and such justification is left as an exercise to the reader.

Stirling's approximation says that $\log(n!) \sim n\log n - n$. Using this, we have

$$\log\left(\frac{\sqrt[n^2]{1!2!\cdots n!}}{n^\alpha}\right) = \frac{1}{n^2}\log(1!2!\cdots n!) - \alpha\log n = \frac{1}{n^2}\sum_{k=1}^{n}\log(k!) - \alpha\log n$$
$$\sim \frac{1}{n^2}\sum_{k=1}^{n}(k\log k - k) - \alpha\log n = \frac{1}{n^2}\sum_{k=1}^{n}(k\log k) - \frac{1}{n^2}\frac{n(n+1)}{2} - \alpha\log n$$

Approximate $\frac{n(n+1)}{2} \sim \frac{n^2}{2}$ and approximate the infinite sum by an integral (and integrate by parts):

$$\sim \frac{1}{n^2}\int_1^n x\log x\,dx - \frac{1}{2} - \alpha\log n = \frac{1}{n^2}\left(\frac{n^2}{2}\log n - \frac{n^2}{4} + \frac{1}{4}\right) - \frac{1}{2} - \alpha\log n$$
$$\sim \frac{1}{2}\log n - \frac{3}{4} - \alpha\log n.$$

Therefore,

$$\lim_{n\to\infty}\log\left(\frac{\sqrt[n^2]{1!2!\cdots n!}}{n^\alpha}\right) = \lim_{n\to\infty}\left[\left(\frac{1}{2}-\alpha\right)\log n - \frac{3}{4}\right],$$

which is finite only when $\frac{1}{2} - \alpha = 0$, in which case $\alpha = \frac{1}{2}$ and the limit evaluates to $-\frac{3}{4}$. Therefore, we wish to compute

$$\lim_{n\to\infty}\frac{\sqrt[n^2]{1!2!\cdots n!}}{n^{1/2}} = \exp\left[\lim_{n\to\infty}\log\left(\frac{\sqrt[n^2]{1!2!\cdots n!}}{n^\alpha}\right)\right] = \boxed{e^{-3/4}}.$$

[1] `http://en.wikipedia.org/wiki/Stirling's_approximation`

10. Find the maximum of

$$\int_0^1 f(x)^3\,dx$$

given the constraints

$$-1 \le f(x) \le 1, \quad \int_0^1 f(x)\,dx = 0.$$

Answer: 1/4

Solution 1: Consider the expression

$$\int_0^1 (f(x)-1)\left(f(x)+\frac{1}{2}\right)^2\,dx.$$

Since $f(x) \le 1$ this expression is less than or equal to 0. Meanwhile, expanding the integrand gives

$$(f(x)-1)\left(f(x)+\frac{1}{2}\right)^2 = f(x)^3 - \frac{3}{4}f(x) - \frac{1}{4},$$

so its integral is

$$\begin{aligned}\int_0^1 (f(x)-1)\left(f(x)-\frac{1}{2}\right)^2\,dx &= \int_0^1 f(x)^3\,dx - \frac{3}{4}\int_0^1 f(x)\,dx - \frac{1}{4}\int_0^1 dx\\ &= \int_0^1 f(x)^3\,dx - \frac{1}{4},\end{aligned}$$

proving that the answer is at most 1/4. This expression is zero when $f(x) = 1$ or $f(x) = -1/2$, so $\boxed{1/4}$ is indeed the maximum value.

Solution 2: Let f_+ and f_- denote the positive and negative part of f respectively. Define $A_+ = \{f > 0\}$ and $A_- = \{f < 0\}$. Then f_+ and f_- are nonzero only on A_+ and A_-. Also the condition on f means

$$0 \le f_+(x), f_-(x) \le 1, \quad \int_0^1 f_+(x)\,dx = \int_0^1 f_-(x)\,dx = s.$$

We will fix s and try to optimize the value $\int f^3 = \int f_+^3 - \int f_-^3$. For the maximum of $\int f_+^3$, we have the inequality $f_+(x)^3 \le f_+(x)$, and integrating it gives

$$\int_0^1 f_+(x)^3\,dx \le \int_0^1 f_+(x)\,dx = s.$$

Equality holds when $f_+(x)$ is either 0 or 1 for all x. For the minimum of $\int f_-^3$, we claim that the minimum occurs when f_- is constant on A_-. From the inequality

$$\frac{a^3+b^3}{2} \ge \left(\frac{a+b}{2}\right)^3$$

for $1 > a, b > 0$, we speculate that the minimum is achieved when the values of f_- are distributed as close as possible. Then if we denote the length of A_- to be l, the minimum occurs when

$f_-(x) = s/l$ for all $x \in A_-$, and the integral will be $l \cdot (s/l)^3 = s^3/l^2$. This can be made rigorous by applying Jensen's inequality

$$\frac{\int_{A_-} \phi(f_-(x))\,dx}{l} \geq \phi\left(\frac{\int_{A_-} f_-(x)\,dx}{l}\right)$$

for the convex function $\phi(t) = t^3$. This gives the minimum as

$$\frac{\int f_-(x)^3\,dx}{l} \geq \frac{s^3}{l^3}, \quad \int f_-(x)^3\,dx \geq \frac{s^3}{l^2}.$$

Meanwhile A_+ should have length greater than s, since f_+ is only nonzero on A_+ and the integral of $f_+ \leq 1$ over A_+ has length at most A_+. So we have $1 - l \geq s$, $l \leq 1 - s$. Now our objective integral is optimized to be the maximum of a single-variable function in s, as follows:

$$\int_0^1 f(x)^3\,dx \leq s - \frac{s^3}{(1-s)^2}.$$

We differentiate this in s to find the maximum. Since

$$\frac{d}{ds}\left(s - \frac{s^3}{(1-s)^2}\right) = \frac{1-3s}{(1-s)^3},$$

its maximum is obtained at $s = 1/3$. Thus finally we can calculate our answer

$$\frac{1}{3} - \frac{(1/3)^3}{(2/3)^2} = \frac{1}{3} - \frac{1}{12} = \boxed{\frac{1}{4}},$$

and equality occurs when $f(x) = 1$ on a set of length $1/3$ and $f(x) = -s/l = -1/2$ elsewhere.