1. Given that $8x + y \leq 17$ and $2x + 7y \leq 13$, compute the maximum possible value of $x + y$.

   Answer: $\mathbf{\frac{88}{27}}$

   Solution: From the two inequalities, it is always the case that $x + y \leq \frac{88}{27}$. We note that this is realizable by the point $\left(\frac{53}{27}, \frac{35}{27}\right)$, so the answer is $\boxed{\frac{88}{27}}$.

2. Evaluate

$$\sum_{n=0}^{\infty} \frac{\left(\frac{-2}{5}\right)^{\lfloor\sqrt{n}\rfloor}}{\sqrt{n} + \sqrt{n+1}}.$$

   Answer: $\mathbf{\frac{5}{7}}$

   Solution: We assume without proof that $\sum_{n=0}^{\infty} \frac{\left(\frac{-2}{5}\right)^{\lfloor\sqrt{n}\rfloor}}{\sqrt{n} + \sqrt{n+1}} = \sum_{n=0}^{\infty} \sum_{i=n^2}^{(n+1)^2-1} \frac{\left(\frac{-2}{5}\right)^{\lfloor\sqrt{i}\rfloor}}{\sqrt{i} + \sqrt{i+1}}$.

   The purpose of this assumption is to group together consecutive terms with the same sign. From there, because $\lfloor\sqrt{i}\rfloor = n$ for $n^2 \leq i \leq (n+1)^2 - 1$, we have $\sum_{i=n^2}^{(n+1)^2-1} \frac{\left(\frac{-2}{5}\right)^{\lfloor\sqrt{i}\rfloor}}{\sqrt{i}+\sqrt{i+1}} = \left(\frac{-2}{5}\right)^n \sum_{i=n^2}^{(n+1)^2-1} \frac{1}{\sqrt{i+1}+\sqrt{i}} = \left(\frac{-2}{5}\right)^n \sum_{i=n^2}^{(n+1)^2-1} \frac{1}{\sqrt{i+1}+\sqrt{i}} \frac{\sqrt{i+1}-\sqrt{i}}{\sqrt{i+1}-\sqrt{i}} = \left(\frac{-2}{5}\right)^n \sum_{i=n^2}^{(n+1)^2-1} \frac{\sqrt{i+1}-\sqrt{i}}{i+1-i} = \left(\frac{-2}{5}\right)^n \sum_{i=n^2}^{(n+1)^2-1} \sqrt{i+1} - \sqrt{i}$ which telescopes to $\left(\frac{-2}{5}\right)^n \left(\sqrt{(n+1)^2 - 1 + 1} - \sqrt{n^2}\right) = \left(\frac{-2}{5}\right)^n (n + 1 - n) = \left(\frac{-2}{5}\right)^n$. So our sum is simply equal to $\sum_{n=0}^{\infty} \left(\frac{-2}{5}\right)^n = \frac{1}{1-\left(\frac{-2}{5}\right)} = \boxed{\frac{5}{7}}$. To formally prove that this grouping is allowed, you can do clever things with the sandwich theorem, but that is up to you.

3. Compute

$$\frac{1}{\sin^2 \frac{\pi}{10}} + \frac{1}{\sin^2 \frac{3\pi}{10}}.$$

   Answer: 12

   Solution: We begin by using the cosine double angle formula to rewrite $\frac{1}{\sin^2 \frac{\pi}{10}} + \frac{1}{\sin^2 \frac{3\pi}{10}} = \frac{1}{\frac{1}{2} - \frac{1}{2}\cos\frac{\pi}{5}} + \frac{1}{\frac{1}{2} - \frac{1}{2}\cos\frac{3\pi}{5}} = 2\left(\frac{2-(\cos\frac{\pi}{5}+\cos\frac{3\pi}{5})}{1-(\cos\frac{\pi}{5}+\cos\frac{3\pi}{5})+\cos\frac{\pi}{5}\cos\frac{3\pi}{5}}\right)$. Simplifying this reduces to computing $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$ and $\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$.

   For the sum of the cosines, the quickest and most intuitive argument (in my opinion) goes as follows. Since the angles involved are multiples of $\frac{\pi}{5}$, we think of the unit circle and a regular pentagon inscribed in it (with vertex at $(-1, 0)$). If we construct vectors from the origin to the vertices, we note that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$ is the sum of the x coordinates of 2 of the vectors. Also, $S = \cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \cos\frac{9\pi}{5} + \cos\frac{7\pi}{5}$ because it is the reflection of those two vectors across the x-axis (and also because this is a basic property of the cosine). Finally, the sum of all 5 of these vectors must be 0- suppose not. Then we may rotate all 5 vectors by $\frac{\pi}{5}$ to get the exact same vector sum, but the only vector which remains the same when rotated by less than a full revolution is the 0 vector. So the sum of all the x coordinates must be 0. Thus $2S + \cos\frac{5\pi}{5} = 2S - 1 = 0$ and $S = \frac{1}{2}$.

For the product, we note that $\cos\frac{3\pi}{5} = -\cos\frac{2\pi}{5}$ so that $\cos\frac{\pi}{5}\cos\frac{3\pi}{5} = -\cos\frac{\pi}{5}\cos\frac{2\pi}{5}\frac{\sin\frac{\pi}{5}}{\sin\frac{\pi}{5}} = -\frac{\frac{1}{2}\sin\frac{2\pi}{5}\cos\frac{2\pi}{5}}{\sin\frac{\pi}{5}} = -\frac{1}{4}\frac{\sin\frac{4\pi}{5}}{\sin\frac{\pi}{5}} = -\frac{1}{4}$.

We may plug both of these in to get $\frac{1}{\sin^2\frac{\pi}{10}} + \frac{1}{\sin^2\frac{3\pi}{10}} = 2\left(\frac{2-\frac{1}{2}}{1-\frac{1}{2}-\frac{1}{4}}\right) = \boxed{12}$.