1. A square $ABCD$ with side length 1 is inscribed in a circle. A smaller square lies in the circle with two vertices lying on segment $\overline{AB}$ and the other two vertices lying on minor arc $\widehat{AB}$. Compute the area of the smaller square.

   **Answer:** $\frac{1}{25}$

   **Solution:** A simple sketch reveals that the side length of the smaller square, $x$, must satisfy:

   $$\left(\frac{x}{2}\right)^2 + \left(x + \frac{1}{2}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2$$

   by the Pythagorean theorem. Thus the area is $x^2 = \boxed{\frac{1}{25}}$.

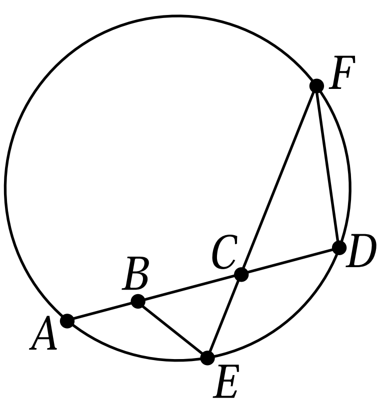
2. Let $ABC$ be a triangle with sides $AB = 19$, $BC = 21$ and $AC = 20$. Let $\omega$ be the incircle of $ABC$ with center $I$. Extend $BI$ so that it intersects $AC$ at $E$. If $\omega$ is tangent to $AC$ at the point $D$, then find the length of $DE$.

   **Answer:** $\frac{1}{2}$

   **Solution:** Since $I$ is the incenter, we know that $BE$ is the angle bisector of $\angle ABC$. By the angle bisector theorem, $\frac{19}{21} = \frac{AB}{AC} = \frac{AE}{EC}$. Plus we have the fact that $AE + CE = AC = 20$, so $AE = \frac{19}{2}$.

   Because $D$ is the point of tangency, we also know that $AD = s - BC$, where $s = \frac{AB + BC + AC}{2}$. Note though that this means that $s = 30$. This implies that $AD = 9$. Finally, $DE = AE - AD = \frac{19}{2} - 9 = \boxed{\frac{1}{2}}$.

3. Circle $O$ has three chords, $AD$, $DF$, and $EF$. Point $E$ lies along the arc $AD$. Point $C$ is the intersection of chords $AD$ and $EF$. Point $B$ lies on segment $AC$ such that $EB = EC = 8$. Given $AB = 6$, $BC = 10$, and $CD = 9$, find $DF$.




   **Answer:** $\frac{9\sqrt{10}}{2}$

   **Solution:** Using power of a point, $AC \cdot CD = EC \cdot CF$ so $CF = 16 \cdot \frac{9}{8} = 18$. Using the Law of Cosines we can find the measure of angle $ECB$, which is congruent to angle $DCF$, $8^2 = 8^2 + 10^2 - 2 \cdot 8 \cdot 10 \cos\theta$. Hence $\cos\theta = 5/8$ and $DF^2 = 9^2 + 18^2 - 2 \cdot 9 \cdot 18 \cos\theta = 405/2$ yielding the answer $DF = \boxed{9\sqrt{10}/2}$.