1. Let $ABCD$ be a quadrilateral with $\angle DAB = \angle ABC = 120°$. If $AB = 3$, $BC = 2$, and $AD = 4$, what is the length of $CD$?

   **Answer:** $\mathbf{\sqrt{39}}$

   **Solution:** Let $E$ be the intersection of rays $DA$ and $CB$. Then $\angle EAB = \angle EBA = 60°$ so $\triangle ABE$ is equilateral. It follows that $EA = EB = 3$. Using Law of Cosines, we find that

   $$CD^2 = EC^2 + ED^2 - 2(EC)(ED)\cos 60° = 5^2 + 7^2 - 5 \cdot 7 = 39$$

   Thus, $CD = \boxed{\sqrt{39}}$.

2. Let $ABCD$ be a rectangle with $AB = 8$ and $BC = 6$. Point $E$ is outside of the rectangle such that $CE = DE$. Point $D$ is reflected over line $AE$ so that its image, $D'$, lies on the interior of the rectangle. Point $D'$ is then reflected over diagonal $AC$, and its image lies on side $AB$. What is the length of $DE$?

   **Answer:** $\mathbf{10\sqrt{5}}$

   **Solution:** Let $D''$ be the image of $D'$ when reflecting across $AC$ so $D''$ is on $AB$. Reflection preserves angles, so let $x = \angle CAD'' = \angle CAD'$. By the same reasoning, $\angle DAE = \angle D'AE$, but $\angle CAD'' + \angle CAD' + \angle D'AE + \angle DAE = \angle BAD = 90°$, so $\angle DAE = 45° - x$.

   Now establish a coordinate plane with $B = (0, 0)$, $A = (0, 8)$, $C = (6, 0)$, and $D = (6, 8)$. The slope of $AE$ is

   $$-\tan(45° - x) = -\frac{\tan 45° - \tan x}{1 + \tan 45° \tan x} = -\frac{1 - \frac{6}{8}}{1 + \frac{6}{8}} = -\frac{1}{7}$$

   so the equation of line $AE$ is $y = 8 - \frac{x}{7}$. But point $E$ must lie on the line $y = 4$, so we have $4 = 8 - \frac{x}{7} \implies x = 28$. This implies that $E = (28, 4)$, so

   $$DE = \sqrt{(28 - 6)^2 + (4 - 8)^2} = \sqrt{22^2 + 4^2} = \sqrt{500} = \boxed{10\sqrt{5}}$$

3. Right triangle $ABC$ with $\angle ABC = 90°$ is inscribed in a circle $\omega_1$ with radius 3. A circle $\omega_2$ tangent to $AB$, $BC$, and $\omega_1$ has radius 2. Compute the area of $\triangle ABC$.

   **Answer: 7**

   **Solution:** Let $O$ and $P$ be the centers of $\omega_1$ and $\omega_2$ respectively. Let $D$ and $E$ be the points of tangency of $\omega_2$ with $AB$ and $BC$ respectively. Then $\angle BDP = \angle BEP = 90°$ and $BD = BE$, so $BDPE$ is a square with side length 2. Thus, $BP = 2\sqrt{2}$. But note that we have $OP = 3 - 2 = 1$ and $OB = 3$ so by the Pythagorean Theorem, $\triangle BPO$ is a right triangle.

   Next, we have that $BP$ bisects $\angle ABC$ so $\angle OBC = 45° - \angle OBP$. Since $OB = OC$, we have $\angle ACB = \angle OBC$. Thus,

   $$\begin{aligned} \sin \angle ACB &= \sin(45° - \angle OBP) \\ &= \sin 45° \cos \angle OBP - \cos 45° \sin \angle OBP \\ &= \frac{\sqrt{2}}{2} \cdot \frac{2\sqrt{2}}{3} - \frac{\sqrt{2}}{2} \cdot \frac{1}{3} \\ &= \frac{4 - \sqrt{2}}{6} \end{aligned}$$

   We then have that $AB = AC \sin \angle ACB = 4 - \sqrt{2}$. Finally, using the Pythagorean Theorem yields $BC = \sqrt{6^2 - (4 - \sqrt{2})^2} = 4 + \sqrt{2}$, so the area of $\triangle ABC$ is $\frac{(4-\sqrt{2})(4+\sqrt{2})}{2} = \boxed{7}$.