1. Given $x + y = 7$, find the value of x that minimizes $4x^2 + 12xy + 9y^2$.

   Answer: 21

   Solution: Factoring,
$$4x^2 + 12xy + 9y^2 = (2x + 3y)^2.$$
   This is minimized when $2x + 3y = 0$. So, $2x + 3(7 - x) = 0$. Thus, $x = 21$.

2. There are real numbers b and c such that the only x-intercept of $8y = x^2 + bx + c$ equals its y-intercept. Compute $b + c$.

   Answer: 48

   Solution: Let the intercept be a. Then the equation may be rewritten as $8y = (x - a)^2$, which equivalently can be written as $y = \frac{1}{8}(x - a)^2$. When $x = 0$, we have $y = \frac{1}{8}a^2 = a$, since we are given that the x-intercept and the y-intercept are equal. Solving for a, we get $a = 8$. Finally, $8y = x^2 - 16x + 64$, so we know that $b = -16$ and $c = 64$, hence $b + c = -16 + 64 = \boxed{48}$.

3. Consider the set of 5 digit numbers $ABCDE$ (with $A \neq 0$) such that $A+B = C$, $B+C = D$, and $C + D = E$. What's the size of this set?

   Answer: 8

   Solution: Observe that $E = D + C = 2C + B = 3B + 2A$ and $E \geq D \geq C \geq A, B$. We proceed by computing the number of digits (A, B) with $A \neq 0$ such that $2A + 3B < 10$, and apply casework on A:

   $A = 1$: B can be 0, 1, or 2

   $A = 2$: B can be 0 or 1

   $A = 3$: B can be 0 or 1

   $A = 4$: B can only be 0

   Notice that once A and B are determined, the other digits are also determined. Hence, there are a total of $\boxed{8}$ such numbers.

4. Let D be the midpoint of BC in $\triangle ABC$. A line perpendicular to D intersects AB at E. If the area of $\triangle ABC$ is four times that of the area of $\triangle BDE$, what is $\angle ACB$ in degrees?

   Answer: 90

   Solution: We have that the area of $\triangle BDE$ is $\frac{1}{2}(BD)(ED)$, and the area of $\triangle ABC$ is $\frac{1}{2}(BC)h = (BD)h$, where h is the height of the altitude from A to BC. Since the area of $\triangle ABC$ is four times that of $\triangle BDE$, we have that $h = 2(ED)$. This is only possible if E is the midpoint of AB. But then $AE = BE = CE$, so E is the circumcenter of $\triangle ABC$. But E lies on AB, so AB is a diameter of the circumcircle, implying that $\angle ACB = \boxed{90^\circ}$.

5. Define the sequence $c_0, c_1, \ldots$ with $c_0 = 2$ and $c_k = 8c_{k-1} + 5$ for $k > 0$. Find $\lim_{k\to\infty} \frac{c_k}{8^k}$.

   Answer: $\frac{19}{7}$

   Solution 1: Notice that $c_{k+1} = 8c_k + 5$, so $c_{k+1} - c_k = 8c_k - 8c_{k-1}$. This gives us a homogenous linear recurrence with characteristic polynomial $x^2 - 9x + 8 = 0$, which has roots $8, 1$. This means that $c_k = a8^k + b1^k$ for some constants a, b. Using $c_0 = 2$ and $c_1 = 21$, we can solve for a and b to find $c_k = \frac{19}{7}8^k - \frac{5}{7}$. Then, the answer immediately follows.

   Solution 2: We can view this sequence in base 8. The zeroth term is 2. To produce the next term, the digit 5 is appended to the end. Thus, the limit approaches 2.5555555.... in base 8, which is $2 + \frac{5}{7} = \boxed{\frac{19}{7}}$ in base 10.

6. Find the maximum possible value of $|\sqrt{n^2+4n+5}-\sqrt{n^2+2n+5}|$.

Answer: $\mathbf{\sqrt{2}}$

Solution: Notice that

$$|\sqrt{n^2+4n+5}-\sqrt{n^2+2n+5}| = |\sqrt{(n+2)^2+(0-1)^2}-\sqrt{(n+1)^2+(0-2)^2}|.$$

If we let P be the point $(n, 0)$, A be the point $(-2, 1)$ and B be the point $(-1, 2)$ on the xy coordinate plane, then the expression above represents the absolute difference between PA and PB. By triangle inequality, $AB \geq PA - PB$ and $AB \geq PB - PA$, so $AB = \boxed{\sqrt{2}}$ is the maximum value. The triangle inequality tells us that this bound is tight when A, B, P are collinear, which happens with $n = -3$.

7. Let $f(x) = \sin^8(x) + \cos^8(x) + \frac{3}{8}\sin^4(2x)$. Let $f^{(n)}(x)$ be the nth derivative of f. What is the largest integer a such that 2^a divides $f^{(2020)}(15°)$?

Answer: 4037

Solution: Note that

$$\begin{aligned}(\sin^2(x)+\cos^2(x))^4 &= \sin^8(x)+\cos^8(x)+4\sin^2(x)\cos^2(x)(\sin^4(x)+\cos^4(x))+6\sin^4(x)\cos^4(x)\\ &= \sin^8(x)+\cos^8(x)+\frac{3}{8}\sin^4(2x)+\sin^2(2x)(1-\frac{1}{2}\sin^2(2x)).\end{aligned}$$

So, we have that

$$\begin{aligned}\sin^8(x)+\cos^8(x)+\frac{3}{8}\sin^4(2x) &= 1-\sin^2(2x)(1-\frac{1}{2}\sin^2(2x))\\ &= 1-\left(\frac{1-\cos(4x)}{2}\right)\left(\frac{3+\cos(4x)}{4}\right)\\ &= \frac{5}{8}+\frac{1}{4}\cos(4x)+\frac{1}{8}\cos^2(4x).\end{aligned}$$

Now $f'(x)$ is

$$-\sin(4x)-\cos(4x)\sin(4x) = -\sin(4x)-\frac{1}{2}\sin(8x).$$

Hence, $f^{(2020)}(x)$ is

$$4^{2019}\cos(4x)+\frac{1}{2}8^{2019}\cos(8x)).$$

Taking $x = 15°$, we have $\cos(4x) = \frac{1}{2}$ and $\cos(8x) = -\frac{1}{2}$. So, we have that

$$f^{(2020)}(15°) = \frac{1}{2}4^{2019}-\frac{1}{4}8^{2019} = 2^{4037}(1-2^{2018}).$$

So, the largest power of two dividing the expression is $\boxed{4037}$.

8. Let $\mathbb{R}^n$ be the set of vectors $(x_1, x_2, \ldots, x_n)$ where $x_1, x_2, \ldots, x_n$ are all real numbers. Let $||(x_1, \ldots, x_n)||$ denote $\sqrt{x_1^2+\ldots+x_n^2}$. Let S be the set in $\mathbb{R}^9$ given by

$$S = \{(x, y, z) : x, y, z \in \mathbb{R}^3, 1 = ||x|| = ||y-x|| = ||z-y||\}$$

If a point (x, y, z) is uniformly at random from S, what is $E[||z||^2]$?

Answer: 3

Solution:

Note that if we write $r_1 = x, r_2 = y - x, r_3 = z - y$, then we have

$$S = \{(r_1, r_1 + r_2, r_1 + r_2 + r_3) : r_1, r_2, r_3 \in \mathbb{R}^3, 1 = ||r_1|| = ||r_2|| = ||r_3||\}.$$

In this case, we have $E[||z||^2] = E[||r_1 + r_2 + r_3||^2]$. Note that if $(r_i)_j$ is the jth coordinate of r_i, then $E[(r_i)_j] = 0$ by symmetry. So by linearity of independence, and the fact that r_1, r_2, r_3 are independent, we have

$$E[||z||^2] = E[(r_1)_1^2 + (r_2)_1^2 + (r_3)_1^2 + ...] + 2E[(r_1)_1(r_2)_1 + (r_1)_1(r_3)_1 + (r_2)_1(r_3)_1 + ...].$$

In this expansion, note that we can group the squared terms by r_i, i.e.

$$E[(r_1)_1^2 + (r_2)_1^2 + (r_3)_1^2 + ...] = E[||r_1||^2] + E[||r_2||^2] + E[||r_3||^2] = 1 + 1 + 1 = 3.$$

The other terms in this expansion are of the form $E[(r_i)_j(r_k)_\ell] = E[(r_i)_j]E[(r_k)_\ell] = 0$. Thus, we conclude that

$$E[||z||^2] = E[||r_1||^2] + E[||r_2||^2] + E[||r_3||^2] = \boxed{3}.$$

9. Let $f(x)$ be the unique integer between 0 and $x-1$, inclusive, that is equivalent modulo x to $\left(\sum_{i=0}^{2} \binom{x-1}{i}((x-1-i)! + i!)\right)$. Let S be the set of primes between 3 and 30, inclusive. Find $\sum_{x \in S} f(x)$.

Answer: 59

Solution: Expanding, we have that

$$\left(\sum_{i=0}^{2} \binom{x-1}{i}((x-1-i)! + i!)\right) = 2(x-1)! + \frac{(x-1)!}{2} + 1 + (x-1) + (x-1)(x-2).$$

By Wilson's, if x is a prime, then $(x-1)! \equiv -1 \mod x$, hence we take the equation mod x to get $-2 - 2^{-1} + 1 - 1 + 2 = -2^{-1}$, where 2^{-1} denotes the multiplicative inverse of 2 mod n. If x is an odd prime, then 2^{-1} is $\frac{x+1}{2}$ and hence $f(x) = \frac{x-1}{2}$. The primes between 3 and 30 are 3, 5, 7, 11, 13, 17, 19, 23, and 29. Therefore, we compute that the desired sum is $\boxed{59}$.

10. In the Cartesian plane, consider a box with vertices $(0,0), (\frac{22}{7}, 0), (0, 24), (\frac{22}{7}, 24)$. We pick an integer a between 1 and 24, inclusive, uniformly at random. We shoot a puck from $(0,0)$ in the direction of $(\frac{22}{7}, a)$ and the puck bounces perfectly around the box (angle in equals angle out, no friction) until it hits one of the four vertices of the box. What is the expected number of times it will hit an edge or vertex of the box, including both when it starts at $(0,0)$ and when it ends at some vertex of the box?

Answer: $\frac{113}{6}$

Solution: More generally, consider a box with vertices $(0,0), (m,0), (0,n), (m,n)$, where $m \neq 0$ and n is a positive integer. We pick an integer a between 1 and n, inclusive, uniformly at random. We will show that it hits the box $\frac{n+a}{\gcd(n,a)}$ times. This formula is sufficient to do casework to calculate the expected value requested in the question.

Consider tiling the entire plane with many translated copies of the box. From this point of view, bouncing around the box just means traveling in a straight line. The puck only stops when it hits one of the vertical lines marking a multiple of m in the x-direction and simultaneously has y-coordinate that's a multiple of n. Let's say it stops after traveling k box-widths. Then, we have that k is the smallest positive integer such that n divides ka, i.e. $k = \frac{n}{\gcd(n,a)}$.

Now, let's count how many times this hits the box. Since it travels k box-widths, it hits the vertical lines $k + 1 = \frac{n}{\gcd(n,a)} + 1$ times. Moreover, it travels $\frac{ak}{n} = \frac{a}{\gcd(n,a)}$ box-heights, so it hits the horizontal lines $\frac{a}{\gcd(n,a)}+1$ times. However, we've over-counted exactly twice, since the puck simultaneously hits vertical and horizontal lines when it hits a vertex, and that happens only at the very beginning and at the very end. Thus, the puck hits the edges $\frac{n+a}{\gcd(n,a)}$ times, as claimed.

Now we need the expectation value of the number of hits. This is

$$\frac{1}{24}\sum_{a=1}^{24}\frac{24+a}{\gcd(24,a)}=\boxed{\frac{113}{6}}$$

11. Sarah is buying school supplies and she has \$2019. She can only buy full packs of each of the following items. A pack of pens is \$4, a pack of pencils is \$3, and any type of notebook or stapler is \$1. Sarah buys at least 1 pack of pencils. She will either buy 1 stapler or no stapler. She will buy at most 3 college-ruled notebooks and at most 2 graph paper notebooks. How many ways can she buy school supplies?

Answer: 4033

Solution: We can create generating functions that describes the number of ways to buy supplies given \$$n$. The pens gives us the generating function $1+x^4+x^8+...=\frac{1}{1-x^4}$ and the pencils give us $x^3+x^6+...=\frac{x^3}{1-x^3}$. Then the stapler gives us a factor of $(1+x)$. Now the notebook conditions give us $1+x+x^2=\frac{1-x^3}{1-x}$ and $1+x+x^2+x^3=\frac{1-x^4}{1-x}$. Multiplying these together, we have

$$\frac{1}{1-x^4}\frac{x^3}{1-x^3}\frac{1+x}{1}\frac{1-x^3}{1-x}\frac{1-x^4}{1-x}=(1-x)\frac{x^3(1+x)}{(1-x)^3}$$

Recognizing (or deriving with derivatives using the generating function $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$) we know that $\frac{x^3(1+x)}{(1-x)^3}=\sum_{n=0}^{\infty}n^2x^{n+2}$. So, we know that

$$(1-x)\sum_{n=0}^{\infty}n^2x^{n+2}=\sum_{n=3}^{\infty}((n-2)^2-(n-3)^2)x^n$$

Finding the coefficient in front of $n=2019$ gives us the number of ways to buy school supplies, which is $2017^2-2016^2=\boxed{4033}$.

12. Let O be the center of the circumcircle of right triangle ABC with $\angle ACB=90^\circ$. Let M be the midpoint of minor arc $\widehat{AC}$ and let N be a point on line BC such that $MN\perp BC$. Let P be the intersection of line AN and the Circle O and let Q be the intersection of line BP and MN. If $QN=2$ and $BN=8$, compute the radius of the Circle O.

Answer: 5

Solution: Let X be the intersection between OM and AC. Since M is the midpoint of arc AC, $\angle MXC=90^\circ$. But we also have $\angle MNC=90^\circ$ and $\angle NCX=90^\circ$, so from quadrilateral $MNCX$ we have that $\angle NMX=90^\circ$. It follows that NM is tangent to Circle O.

Since P lies on Circle O, $\angle APB=90^\circ$. Then $\angle QPN=\angle APB=90^\circ=\angle QNB$ and $\angle NQP=\angle BQN$, we have that $\triangle NPQ\sim\triangle BNQ$. Then $\frac{QN}{QB}=\frac{QP}{QN}\implies QP(QB)=QN^2$. But by Power of a Point, $QP(QB)=QM^2$, so $QM=QN=2$. Thus, $MN=4$.

From $\angle MNB=90^\circ=\angle NMO$, we have $OM||BN$, so $BOMN$ is a trapezoid. Let Y be the perpendicular drawn from O to BC and let r be the radius of Circle O. Then we have $BY=BN-NY=BN-MO=8-r$. Also, $OY=MN=4$. By the Pythagorean Theorem on right triangle BOY, we have $r^2=(8-r)^2+4^2$. Solving for r, we get $r=\boxed{5}$.

13. Reduce the following expression to a simplified rational:

$$\frac{1}{1-\cos\frac{\pi}{9}}+\frac{1}{1-\cos\frac{5\pi}{9}}+\frac{1}{1-\cos\frac{7\pi}{9}}$$

Answer: $\boxed{\mathbf{18}}$

Solution: Rewrite the expression as

$$\sum_{i=1}^{3}\frac{1}{1-\cos\theta_i}.$$

The biggest annoyance is that θ_i all have a 9 in the denominator. If only the angles were all multiplied by 3 – the problem would be much easier! In fact, $\cos(3\theta_i)=\frac{1}{2}$ for all i. Then, using the triple angle formula, $4\cos^3\theta_i-3\cos\theta_i=\frac{1}{2}$. Let $x_i=\cos\theta_i$. Then we wish to compute

$$\sum_{i=1}^{3}\frac{1}{1-x_i}$$

given that

$$f(x)=4(x-x_1)(x-x_2)(x-x_3)=4x^3-3x-\frac{1}{2}.$$

The next obstacle is that the original expression is in terms of $1-x_i$. Let's take a look at $f(1-x)$ whose roots are $r_i=1-x_i$:

$$f(1-x)=4(1-x)^3-3(1-x)-\frac{1}{2}\to x^3-3x^2+\frac{9}{4}x-\frac{1}{8}=(x-r_1)(x-r_2)(x-r_3),$$

where I normalize the rightmost equation by the leading coefficient. Using Vieta's formulas, we can finally compute

$$\sum_{i=1}^{3}\frac{1}{1-\cos\theta_i}=\sum_{i=1}^{3}\frac{1}{r_i}=\frac{1}{r_1r_2r_3}\cdot(r_1r_2+r_2r_3+r_3r_1)=\frac{1}{(1/8)}\cdot(9/4)=\boxed{18}.$$

14. Compute the following integral:

$$\int_0^\infty \log\left(1+e^{-t}\right)dt.$$

Answer: $\boxed{\boldsymbol{\frac{\pi^2}{12}}}$

Solution: The following Taylor series is well known:

$$\log(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}.$$

It converges for $x \in (-1, 1]$. We see that $e^{-t} \in (0, 1]$ for $t \in [0, \infty)$, so we can safely use the Taylor expansion in the integral. Glossing over why the integral and sum can be interchanged, we find the following:

$$\int_0^\infty \log\left(1+e^{-t}\right) dt = \int_0^\infty \sum_{n=1}^\infty \frac{(-1)^{n-1}e^{-nt}}{n} dt = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} \int_0^\infty e^{-nt} dt =$$

$$= \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} \left(\frac{-e^{-nt}}{n}\right)\Bigg|_0^\infty = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2}.$$

This is pretty similar to the well known $S = \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$. The alternating version of this sum is just the original sum minus twice the sum of the even squares. In other words,

$$\int_0^\infty \log\left(1+e^{-t}\right) dt = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2} = \sum_{n=1}^\infty \frac{1}{n^2} - 2\sum_{n=1}^\infty \frac{1}{(2n)^2} = S - \frac{S}{2} = \boxed{\frac{\pi^2}{12}}.$$

15. Define $f(n)$ to be the maximum possible least-common-multiple of any sequence of positive integers which sum to n. Find the sum of all possible odd $f(n)$.

Answer: 124

Solution: It is certainly possible to get this answer with a mix of computation and intuition. Here is a formal proof.

For convention, let us call a sequence $(s_1, s_2, \ldots, s_k)$ of positive integers "optimal for n" if $\sum_i s_i = n$ and $LCM(s_1, s_2, \ldots, s_k) = f(n)$.

We notice a few things.

- There exists an optimal arrangement $(s_1, s_2, \ldots, s_k)$ for n with $GCD(s_1, s_2, \ldots, s_k) = 1$. Say that $(s_1, s_2, \ldots, s_k)$ is an optimal arrangement and assume $\exists i, j$ with $s_i = ab$ and $s_j = ac$, with a being some positive integer greater than 1 and b and c relatively prime. Then we can replace s_j in the list with $(a, 1, 1, \ldots, 1)$ (with 1 repeated $(ac) - (a)$ times). We can continue doing this on any pairs s_i, s_j that are not relatively prime until we have a longer list with elements pairwise relatively prime. Note the LCM is the same so this arrangement is also optimal, and all elements are pairwise relatively prime. Thus, we will continue under the assumption that all optimal arrangements considered consist only of elements which are pairwise relatively prime. (Note, all positive integers are relatively prime to 1).
- There exists an optimal arrangement $(s_1, s_2, \ldots, s_k)$ for n with all elements powers of primes, or 1. If we have $s_i = p_1^{e_1} p_2^{e_2}$, we can simply replace it with the sequence $(p_1^{e_1}, p_2^{e_2}, 1, 1, \ldots, 1)$, where there are $p_1^{e_1} p_2^{e_2} - p_1^{e_1} - p_2^{e_2}$ repetitions of 1 (it is easy to show that this is nonnegative). We can adopt a similar argument on any elements that are the product of multiple prime powers. Thus, we can repeat this process until we have a list consisting of (pairwise relatively prime) prime powers and 1's.
- Consider an optimal arrangement for n, $(s_1, s_2, \ldots, s_k)$ with all elements pairwise relatively prime prime powers and 1's. Consider the smallest p such that an element p^a appears in the list, with $a > 0$. We show $p < 5$. If $p \geq 5$ and $a = 1$, we can replace p with $p - 2$ and 2. Obviously $2 \times (p - 2) > p$. Note also that 2 and $p - 2$ must be pairwise relatively prime to one another because p is prime, and to every other element in the list, because all other elements in the list are prime powers derived from larger primes. So the LCM of 2, $p - 2$, and the rest of the list is indeed the product of all of

these numbers. Thus because $2 \times (p-2) > p$ this is a contradiction of the fact that the list is optimal for n. Consider now if $a > 1$. Then, similarly, we can replace p^a by the sequence $(p^{a-1}, p-2, 2, 1, 1, \ldots, 1)$ where there are $p^a - p^{a-1} - p$ 1's. Note that $p^a - p^{a-1} - p = p^{a-1}(p-1) - p > 0$ for $a > 1$ and any p. The argument regarding relatively primeness is the same as with $a = 1$.

- From here on, let us consider an optimal arrangement that does not contain any powers of 2 (this is the interesting case - when $f(n)$ is odd). Now we show that all primes in the list must be raised to the first power. Consider some prime p such that p^a is in the list, $a > 1$. Clearly, there is a power of 2 (call it 2^x) between p and $2p$. In this case we could replace p^a by the sequence $(p^{a-1}, 2^x, 1, 1, 1\ldots, 1)$ where there are $p^a - p^{a-1} - 2^x$ 1's. It is easy to see $p^a - p^{a-1} - 2^x \geq 0$, because $p^a - p^{a-1} - 2^x > p^a - p^{a-1} - 2p = p^{a-1}(p-1) - 2p$ and clearly $p^{a-1}(p-1) \geq 2p$, because $p - 1 \geq 2$ by assumption p is odd and $p^{a-1} \geq p$ by assumption $a > 1$. We are now contributing $p^{a-1}2^x$ to our LCM instead of p^a, but $p^{a-1}2^x > p^a$ which is a contradiction that the arrangement is optimal. Thus all primes must be raised to the first power.
- Given that we have an optimal arrangement where all elements of the list are odd, we can prove that primes (recall we also proved that they are all raised to power 1) must appear in consecutive order. Say that primes p_1 and p_3 appear in the list, and they are not consecutive primes (i.e., there is at least 1 prime p_2 with $p_1 < p_2 < p_3$). By Bertrand's Postulate, we can choose some p_2 with $p_1 < p_2 < p_3$ and $p_2 > p_3/2$. Then, we can replace p_3 by the sequence $(p_2, p_3 - p_2)$. Because $p_3 - p_2$ is even, it contributes a factor of 2 to the LCM, and p_2 contributes a factor of p_2. Thus, we have replaced the factor of p_3 in the LCM with (at least) $2p_2$, which is larger than p_3. This is a contradiction that the arrangement is optimal, thus in an optimal arrangement with all terms odd, no two primes can appear in the list without all of the primes between them also appearing in the list.
- Taking all of our results about optimal arrangements and odd optimal arrangements together, we find that every odd optimal arrangement is in the form

$$(1, 1, \ldots, 1, 3, 5, \ldots),$$

continuing up the odd primes consecutively. Our final (and most important) result is that the number 11 cannot be included in an odd optimal arrangement. We already proved that all odd optimal arrangements must contain 3 (because the lowest prime must be smaller than 5). So then consider a list that contains 3 and 11. We can replace 3 and 11 with 1, 4, and 9. Note that these replacements leave the optimal arrangement relatively prime, so we can again consider the LCM as the product of elements. And note that $9 \times 4 > 3 \times 11$. Thus we have improved the arrangement, so it is not optimal. So no odd arrangement that contains 11 is optimal. Recall that we proved before that in an odd optimal arrangement, all prime factors must appear consecutively; thus, if an odd optimal arrangement does not contain 11, it cannot contain 13, 17, etc. Thus no odd optimal arrangement contains any terms 11 or greater.
- Finally, it is obvious that an odd optimal arrangement cannot contain more than a single 1 - else, we could replace it with a 2 and increase the LCM. Moreover, if we have 3 in our optimal odd arrangement, then we must not have any 1's: if we have a 1, then we can replace 3 and 1 with 4, which is relatively prime to all odd primes and thus increases the LCM.
- Thus the only odd arrangements that we have not ruled out from being optimal are

$$(1), (3), (3, 5), (3, 5, 7).$$

With computation, we can see that these are all optimal and that they yield distinct $f(n)$ of $1, 3, 15, 105$. Therefore, the answer is $\boxed{124}$.