1. Katy writes down an odd composite positive integer less than 1000. Katy then generates a new integer by reversing the digits of her initial number. The new number is a multiple of 25 and is also less than her initial number. What was the initial number that Katy wrote down?

   Answer: 573

   Solution: Because the reversed number is a multiple of 25, it must end in either 00, 25, 50, or 75. However, because the leading digit of the initial number cannot be a 0, the reversed number must end in 25 or 75. Therefore, the initial number must start with either the digits 52 or 57. Because the reversed number is less than the initial number, the last digit of the initial number must be less than 5. Furthermore, since the initial number is odd, the last digit must be either 1 or 3. Therefore, the four possible values of Katy's initial number are 521, 523, 571, and 573. Of these, only 573 is composite as it is equal to $3 \cdot 191$. Therefore, Katy's initial number was 573.

2. Find the probability that a randomly selected divisor of 20! is a multiple of 2000.

   Answer: $\mathbf{\frac{6}{19}}$

   Solution: The prime factorization of 2000 is $2000 = 2^4 \cdot 5^3$. So we essentially need to find the probability that a randomly selected divisor of 20! has both a factor of 2^4 and of 5^3. The prime factorization of:

   $$20! = 2^{18} \cdot 3^8 \cdot 5^4 \cdots$$

   so a randomly selected divisor of 20! is in the form $2^a \cdot 5^b \cdot k$ where k is an integer with no factor of 2 or 5, a is chosen randomly from {0, 1, ..., 18} and b is chosen randomly from {0, 1, ... , 4}. There are 15 successful choices for a of a possible 19 and 6 successful choices for 2 out of a possible 5. Hence our probability is $\frac{15}{19} \cdot \frac{2}{5} = \frac{6}{19}$.

3. Let $A(n) = \sum_{i=1}^{n} \lceil \frac{n}{i} \rceil$ and $B(n) = \sum_{i=1}^{n} \lfloor \frac{n}{i} \rfloor$. Compute $A(2020) - B(2020)$.

   Answer: 2008

   Solution:

   $$A(2020) - B(2020) = \sum_{i=1}^{2020} \left\lceil \frac{2020}{i} \right\rceil - \left\lfloor \frac{2020}{i} \right\rfloor,$$

   where each term $\lceil \frac{2020}{i} \rceil - \lfloor \frac{2020}{i} \rfloor$ is 0 if $\frac{2020}{i}$ is an integer and 1 if $\frac{2020}{i}$ is not an integer. Since $\frac{2020}{i}$ is an integer only when i is a factor of 2020, the value of $A(2020) - B(2020)$ is equal to $2020 - \tau(2020)$. Since $2020 = 2^2 \times 5^1 \times 101^1$, $\tau(2020)$, the number of factors of 2020, is equal to $3 \times 2 \times 2 = 12$. Therefore, $A(2020) - B(2020) = 2020 - 12 = 2008$.

4. What is the smallest positive multiple of 2020 that has all distinct digits?

   Answer: 351480

   Solution: A number is divisible by 2020 if and only if it is divisible by both 20 and 101. For a number to be divisible by 20, it must end in either 00, 20, 40, 60, or 80. Because $101 \equiv -1 \pmod{100}$, a number is divisible by 101 if and only if the sum of alternating two-digit blocks is divisible by 101.

   Note that there are no four-digit numbers that fit the condition as 2020, 4040, 6060, and 8080 each have two 0s as digits. Therefore, the smallest number must have at least 5 digits. In order for a five-digit number $ABCDE$ to be divisible by 2020, D must be even, E must be 0, and $A + DE - BC$ must be a multiple of 101. Because $A + DE \leq 9 + 80 < 101$, $A + DE - BC$ can only be a multiple of 101 if $A + DE - BC = 0$, However, since $E = 0$, A must be equal to C, meaning the number does not have all distinct digits. Therefore, the smallest number must be have at least 6 digits.

Similarly, for a six-digit number $ABCDEF$ to be divisible by 2020, $AB + EF - CD$ must be a multiple of 101 with E being even and F being 0. Since $AB + EF \leq 99 + 80 < 202$, $AB + EF - CD$ must be either equal to 0 or 101. However, if $AB + EF - CD = 0$, then similar to the five-digit case, B would equal D. Therefore, $AB + EF - CD = 101$. Note that the value of $ABCDEF$ is minimized when $AB = 101 + CD - EF$ is minimized. For AB to be minimized, maximize EF to be equal to 80, meaning that $AB = 21 + CD$. Because $F = 0$, the smallest possible value of C is 1, implying that $A = 3$. Furthermore, the smallest value of D is 4 as D cannot be 2 as that would imply $A = B = 3$. Therefore, the minimum occurs when $ABCDEF = 351480$.

Note that 351480 is indeed the minimum as if $EF \neq 80$, then $EF \leq 60$. However, this would imply $AB = 101 + CD - EF \geq 101 + CD - 60 = 41 + CD > 35$, which would result in a larger six-digit number.

5. Find the smallest integer value of n such that

$$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{n\text{ 2's}} \geq 16^{16^{16^{16}}}.$$

Answer: 7

Solution: The natural first step is to start by taking the base-2 logarithm of the right hand side:

$$\log_2\left(16^{16^{16^{16}}}\right) = 4 \cdot 16^{16^{16}}.$$

We can do this again:

$$\log_2\left(4 \cdot 16^{16^{16}}\right) = \log_2 4 + \log_2\left(16^{16^{16}}\right) = 2 + 4 \cdot 16^{16}.$$

Note that $2 + 4 \cdot 16^{16} = 2 + 4 \cdot 2^{64} = 2 + 2^{66}$. On the other hand,

$$2^{2^{2^{2}}} = 2^{16} < 2 + 2^{66} < 2^{2^{16}} = 2^{2^{2^{2^{2}}}}.$$

Hence

$$2^{2^{2^{2^{2^{2}}}}} < 16^{16^{16^{16}}} < 2^{2^{2^{2^{2^{2^{2}}}}}},$$

and the answer is $n = 7$.

6. William has a bag of white, milk, and dark chocolate bars. Each minute he reaches into the bag, selects a chocolate bar at random, and eats it. Given that there are 17 milk chocolate bars, 12 dark chocolate bars, and 19 white chocolate bars, what is the probability that William runs out of milk chocolate bars first and dark chocolate bars second?

Answer:

$$\mathbf{\frac{19}{116}}$$

Solution: We can equivalently find the probability that William runs out of white chocolate bars LAST and dark chocolate bars second to last. Equivalently this is the probability that the last remaining bar is white chocolate, and INDEPENDENTLY among the milk and dark chocolate bars, the last one remaining is dark. The probability that the last remaining bar is white is:

$$\frac{19}{17+12+19} = \frac{19}{48}$$

because it is just the probability of randomly selecting a white chocolate bar from all of the bars in the bag. The probability that among the milk and dark chocolate bars, the last remaining is dark is equivalently the probability of randomly selecting a dark chocolate bar from among the dark and milk which is:

$$\frac{12}{17+12} = \frac{12}{29}$$

Multiplying our independent probabilities we get:

$$\frac{19}{48} \cdot \frac{12}{29} = \frac{19}{116}$$

7. A rook is on a chess board with 8 rows and 8 columns. The rows are numbered 1, 2, ..., 8 and the columns are lettered a, b, ..., h. The rook begins at a1 (the square in both row 1 and column a). Every minute, the rook randomly moves to a different square either in the same row or the same column. The rook continues to move until it arrives a square in either row 8 or column h. After infinite time, what is the probability the rook ends at a8?

   Answer: $\frac{1}{6}$

   Solution: Observe that there are are a total of 14 possible squares that the rook can end at as the rook can never reach h8. By symmetry, if the rook is at square xY where x represents a column from a to g and Y represents a row from 1 to 7, Rook has equal probability at x8 and hY. We will denote this probability as x. Similarly, the rook has an equal probability of ending at each of the other 12 reachable squares in column h or row 8. We will denote this probability as y. This gives us the equation $2x + 12y = 1$. Furthermore, note that x is the probability that the rook ends at a8 when the rook starts at a1.

   Now consider the probability that the rook ends at a8 after the first move. The rook has 14 possible first moves (which can be split into 4 cases):

   (a) The rook moves directly to a8 and stops. This occurs with probability $\frac{1}{14}$.

   (b) The rook moves to one of a2, a3, ..., a7. This occurs with probability $\frac{6}{14}$ and from each these positions, the rook has probability x of ending at a8.

   (c) The rook moves to one of b1, c1, ..., g1. This occurs with probability $\frac{6}{14}$ and from each these positions, the rook has probability y of ending at a8.

   (d) The rook moves directly to h1, stops and cannot reach a8. This occurs with probability $\frac{1}{14}$.

   Together we have $x = \frac{1}{14} \cdot 1 + \frac{6}{14} \cdot x + \frac{6}{14} \cdot y + 0$. Solving with our earlier equation, we get $x = \frac{1}{6}$.

8. Suppose Joey is at the origin and wants to walk to the point $(20, 20)$. At each lattice point, Joey has three possible options. He can either travel 1 lattice point to the right, 1 lattice point above, or diagonally to the lattice point 1 above and 1 to the right. If none of the lattice points

Joey reaches have their coordinates sum to a multiple of 3 (excluding his starting point), how many different paths can Joey take to get to $(20, 20)$?

Answer: 4356

Solution: Notice that the sum of the coordinates will either increase by one if he travels directly upwards or directly to the right or increase by two if he travels diagonally. Therefore, when the sum of the coordinates is congruent to 2 mod 3, he must travel diagonally and when the sum of the coordinates is congruent to 1 mod 3, he must travel directly upwards or directly to the right. If we denote travelling diagonally with the letter D and travelling either upwards or to the right with the letter S, there are two possible combinations for Joey's path.

If Joey's first step is diagonal, he must take the path $D \underbrace{DS}_{\text{12 times}} D$, where 6 of the S steps are travelling to the right and the other 6 of the S steps are travelling upwards. There are $\binom{12}{6} = 924$ possible paths that fit this condition. If Joey's first step is not diagonal, he must take the path $SS \underbrace{DS}_{\text{12 times}} D$, where 7 of the S steps are travelling to the right and the other 7 of the S steps are travelling upwards. There are $\binom{14}{7} = 3432$ possible paths that fit this condition. Therefore, there are a total of $\binom{12}{6} + \binom{14}{7} = 924 + 3432 = 4356$ possible paths Joey can take.

9. Elena and Mina are making volleyball teams for a tournament, so they find 15 classmates and have them stand in a line from tallest to shortest. They each select six students, such that no two students on the same team stood next to each other in line. How many ways are there to choose teams?

Answer: 2570

Solution: Case 0: 0 parity swaps (EMEMEMEMEMEM)

In this case, we know the ordering is fixed as EMEMEMEMEMEM, so there are simply $\binom{15}{3} = 455$ ways to insert 3 unchosen players into this ordering.

Case 1: 1 parity swap (EMEMEM|MEMEME)

In this case, note that the size of the two sections we divide our chosen player into must both be even, as an odd section contains one extra player of a given team, and two adjacent odd sections will therefore have 2 extra players from that team, by parity (EX: EME|EMEMEMEME has 2 more E players than M players). There are 5 ways to divide our interval of length 12 into two nonempty even sections, using an unchosen player, and if the remaining two unchosen players are not placed adjacent to one another, they can be inserted in $\frac{13 \cdot 12}{2} = 78$ ways (note that the choice of placing another unchosen player immediately before or after our original unchosen player is indistinguishable, meaning there are only 13 distinguishable spots to begin with). If they are placed adjacent to one another, this adds 13 possibilities, giving a total of $5 \cdot (78 + 13) = 455$.

Case 2: 2 parity swaps (EME|EMEM|MEMEM)

In this case, we either divide our 12 person roster into 3 even length intervals, or an odd, even, odd, in that order (our only condition on odd length intervals is that we couldn't put them adjacent to one another). The number of ways to have 3 nonempty even intervals summing to 12 is $\binom{3+2}{2} = 10$ by stars and bars, and the number of ways to have an odd, even, odd is $\binom{4+2}{2} = 15$, as it is equivalent to picking 4 (possibly empty) even intervals summing to 8, after we subtract 1 from each odd and 2 from the even interval. In all cases, we can again insert the final unchosen player in 13 ways, giving a total of $(10 + 15) \cdot 13 = 325$.

Case 3: 3 parity swaps (EME|EMEM|MEM|EM)

In this case, our options for interval lengths are either 4 evens, or one of {even, odd, even, odd}, {odd, even, odd, even}. This is because we must have an even number of odd length intervals (since the total, 12, is even), we cannot have any two adjacent, and we also cannot have them be distance 2 apart, because this actually causes the same parity issue as having two odd intervals adjacent (EX: EME|EMEM|EM|EME has 2 more E players than M players). There are $\binom{5}{3} = 10$ ways to have 4 even intervals summing to 12, and $\binom{6}{3} = 20$ ways for each of the alternating cases, again by stars and bars, so we have a total of $10 + 2 \cdot 20 = 50$.

In total, this gives the number of choices for teams as $2 \cdot (455 + 455 + 325 + 50) = 2570$.

10. Suppose n is a product of three primes p_1, p_2, p_3 where $p_1 < p_2 < p_3$ and p_1 is a two-digit integer. If $n - 1$ is a perfect square, compute the smallest possible value of n.

Answer: 30277

Solution: Define k to be the positive integer such that $n - 1 = k^2$. Since $k^2 + 1 = p_1p_2p_3$, it follows that $k^2 \equiv -1 \pmod{p_1}$, $k^2 \equiv -1 \pmod{p_2}$, and $k^2 \equiv -1 \pmod{p_3}$. Because -1 is only a quadratic residue of 2 and odd primes of the form $4x + 1$, the congruence $k^2 \equiv -1 \pmod{p}$ has solutions if and only if prime p is equal to 2 or congruent to 1 mod 4. Therefore, it follows that p_1, p_2, p_3 are primes congruent to 1 mod 4.

Furthermore, for any prime $p < p_1$, it must follow that $k^2 \not\equiv -1 \pmod{p}$. Because p_1 is a two-digit integer, the primes $p < 10 \le p_1$ must be checked. Note that of these primes, only 2 and 5 must be checked as there are no solutions to the congruence when p is 3 or 7. Because $(\pm 1)^2 \equiv -1 \pmod{2}$ and $(\pm 2)^2 \equiv -1 \pmod{5}$, it follows that $k \not\equiv 1 \pmod{2}$ and $k \not\equiv \pm 2 \pmod{5}$. By the Chinese Remainder Theorem, k must be congruent to either 0, 4, or 6 in mod 10.

To find a solution, consider the case where p_1 and p_2 are minimized: $p_1 = 13$, and $p_2 = 17$. Since $(\pm 5)^2 \equiv -1 \pmod{13}$ and $(\pm 4)^2 \equiv -1 \pmod{17}$, it follows that $k \equiv \pm 5 \equiv \pm 8 \pmod{13}$ and $k \equiv \pm 4 \equiv \pm 13 \pmod{17}$. By the Chinese Remainder Theorem, there are four possible values of k mod 221.

To find these values, note that $k \equiv \pm 13 \pmod{17}$ implies that k must be in the form $17x \pm 13$. Taking this mod 13, it follows that $k = 17x \pm 13 \equiv 4x \equiv \pm 8 \pmod{13}$. Therefore, x must be ± 2, meaning that the possible values of $k \pmod{221}$ are generated by the values of $17(\pm 2) \pm 13$. This yields 21, 47, 174, and 200 as the four possible values of k mod 221.

Since k must be congruent to either 0, 4, or 6 in mod 10, the smallest possible value of k in this case is when $k = 174$. In this case, $n = 174^2 + 1 = (13)(17)(137) = 30277$ where 137 is prime. Therefore, $n = 30277$ satisfies the given conditions. To prove that this value of n is the smallest, it suffices to show that no $k < 174$ yields an n that fits the given conditions.

First, consider the maximum possible value of p_2. Because $p_1p_2^2 < p_1p_2p_3 < (13)(17)(137)$ and $p_1 \ge 13$, it follows that $p_2^2 < (17)(137) = 2329$. Since $41^2 = 1681 < 2329 < 53^2 = 2809$, the maximum possible value of p_2 is 41. Therefore, p_1 and p_2 must belong to the set of primes $\{13, 17, 29, 37, 41\}$. For each of these primes, list the possible values of $k < 174$ that satisfy the congruence $k^2 \equiv -1 \pmod{p}$.

p	$k^2 \equiv -1 \pmod{p}$	Even $k < 174$	$k \equiv 0, 4, 6 \pmod{10}$
13	$k \equiv \pm 5 \pmod{13}$	$8, 18, 34, 44, 60, 70, 86, 96, 112, 122, 138, 148, 164$	$34, 44, 60, 70, 86, 96, 112, 164$
17	$k \equiv \pm 4 \pmod{17}$	$30, 38, 64, 72, 98, 106, 132, 140, 168$	$30, 64, 106, 140$
29	$k \equiv \pm 12 \pmod{29}$	$46, 70, 104, 128, 162$	$46, 70, 104$
37	$k \equiv \pm 6 \pmod{37}$	$68, 80, 142, 154$	$80, 154$
41	$k \equiv \pm 9 \pmod{41}$	$32, 50, 124, 132$	$50, 124$

The only value that appears at least twice in the final column is 70. However, if $k = 70$, then $n = 70^2 + 1 = 4901 = (13)(29)(13)$. Because the three prime factors must be distinct, $k = 70$ does not yield an n that fits the given conditions. Therefore, the smallest possible value of n is 30277, which occurs when $k = 174$.