1. Find the remainder when x^6 is divided by $x^2 - 3x + 2$.

   Answer: $63x - 62$

   Solution: Let $q(x)$ be the quotient and $r(x)$ be the remainder. Since $x^2 - 3x + 2$ is quadratic, we know $r(x) = ax + b$. We can then write

$$x^6 = q(x)(x^2 - 3x + 2) + ax + b.$$

   The roots of $x^2 - 3x + 2$ are $1, 2$, so plugging these in we have $1^6 = a + b$ and $2^6 = 2a + b$. Solving this system of equations gives us $a = 63$ and $b = -62$. Thus, the remainder is $r(x) = \boxed{63x - 62}$.

2. Compute the sum of possible integers such that $x^4 + 6x^3 + 11x^2 + 3x + 16$ is a square number.

   Answer: 2

   Solution: We claim that $x = 10$ is the only solution. We will use the fact that for $|n| > |m|$, $n^2 - m^2 \geq 2n - 1$. Consider that the polynomial is $(x^2 + 3x + 1)^2 - 3(x - 5)$. We clearly have a solution at $x = 5$. Then, if y is the root of the square, $(x^2 + 3x + 1)^2 - 3(x - 5) = y^2$. Now we split into cases. If $3(x - 5) > 0$, (i.e. $x > 5$), then $3(x - 5) = (x^2 + 3x + 1)^2 - y^2 \geq 2|x^2 + 3x + 1| - 1$. For $x \geq 5$, we can see that this is false and there are no solutions for $x \geq 5$. Then for $x \leq 5$, we have that $y \geq x^2 + 3x + 1$ and hence $3(5 - x) \geq 2|x^2 + 3x + 1| - 1$ again, we will not hold for $x < -5$. Then we can test all of the intermediate values to see that only $x = 5, 0, -3$ holds. So, we have a sum of $\boxed{2}$.

3. Suppose $f(x) = \sqrt{x^2 - 102x + 2018}$. Let A and B be the smallest integer values of the function that can be derived from integer inputs. Given $A < B$, find A and B.

   Answer: $A = 21$, $B = 291$

   Solution: If $\sqrt{x^2 - 102x + 2018}$ is an integer, then $x^2 - 102x + 2018 = m^2$ for some positive integer m. The integers A and B are the two smallest possible values for m. Completing the square, we have the following equation:

$$(x - 51)^2 + (2018 - 51^2) = m^2 \implies (x - 51)^2 - m^2 = 583$$

   The left expression is a difference of squares, so $((x - 51) + m)((x - 51) - m) = 583$. Since $583 = 11 \times 53$ has 4 factors, the positive difference between the factors, which is represented by $((x - 51) + m) - ((x - 51) - m) = 2m$, is either $53 - 11 = 42$ or $583 - 1 = 582$. Therefore $m = 21, 291 \implies A = 21, B = 291$.

4. Let x and y be complex numbers such that $x^2 + y^2 = 31$ and $x^3 + y^3 = 154$. Find the maximum possible real value of $x + y$.

   Answer: 7

   Solution: Let $a = x + y$, $b = xy$. We have:

$$a^2 = x^2 + 2xy + y^2 = 31 + 2b$$

$$a^3 = x^3 + 3x^2y + 3xy^2 + y^3 = 154 + 3ab$$

   From here, we get $b = \frac{a^2 - 31}{2}$, and substituting this in gives:

$$a^3 = 154 + \frac{3}{2}a^3 - \frac{93}{2}a$$

$$a^3 - 93a + 308 = 0$$

We can see that 4 is a root of this cubic, so we can factorize it completely into:

$$(a-4)(a-7)(a+11) = 0$$

We want the maximum value of a, so our answer is $\boxed{7}$.

5. The function $y = x^2$ does not include the point $(5, 0)$. Let θ be the absolute value of the smallest angle the curve needs to be rotated around the origin so that it includes $(5, 0)$?. Find $\tan(\theta)$

Answer: $\sqrt{\frac{-1+\sqrt{101}}{2}}$

Solution: When Abel drives due east along the Cartesian plane, he begins at $(0, 0)$ and travels 1 mile east and 1 mile left (north/positive y) to the point $(1, 1)$. Then, he travels 1 more mile east and 3 miles north, to the point $(2, 4)$. Similarly, he travels 1 more mile east and 5 miles north, to the point $(3, 9)$. This process can be repeated indefinitely, but ultimately shows that Abel drives along the curve $f(x) = x^2$ relative to his starting direction (east).

We know that Abel wants to eventually reach five miles away from his starting location. Therefore, we need to find a point on the curve that is five miles from $(0, 0)$.

$$\sqrt{(x-0)^2 + (x^2-0)^2} = 5$$
$$\sqrt{x^2 + x^4} = 5$$
$$x^2 + x^4 = 25$$

Replacing x^2 with $z > 0$ for simplicity,

$$z^2 + z - 25 = 0$$
$$z = \frac{1 \pm \sqrt{1+100}}{2}$$

Because $z > 0$,

$$z = \frac{1+\sqrt{101}}{2} \Rightarrow x = \sqrt{\frac{1+\sqrt{101}}{2}}$$

Therefore, the value of x where Abel is five miles from where he started is the value above. Abel's y position for this x value is simply the square of the value above. Therefore, if Abel starts by heading due east, he ends up at an angle of

$$\arctan \frac{\frac{1+\sqrt{101}}{2}}{\sqrt{\frac{1+\sqrt{101}}{2}}} = \arctan \sqrt{\frac{1+\sqrt{101}}{2}}$$

relative to the horizontal.

If he begins his journey heading at the angle $-\arctan\sqrt{\frac{1+\sqrt{101}}{2}}$ relative to horizontal positive x axis (east), he will arrive at his intended destination.

$\boxed{\arctan\sqrt{\frac{1+\sqrt{101}}{2}}}$. Other acceptable answers include anything equivalent, perhaps using arcsin or arccos.

6. The polynomial $1 - 2x + 4x^2 - 8x^3 + ... + 2^{20}x^{20} - 2^{21}x^{21}$ can be expressed as $c_0 + c_1 y + ... + c_{20}y^{20} + c_{21}y^{21}$ where $y = x + \frac{1}{2}$. Find c_2.

   Answer: 6160

   Solution: We have $1 - 2x + 4x^2 - 8x^3 + ... + 2^{20}x^{20} - 2^{21}x^{21} = \frac{1-(2x)^{22}}{1+2x}$. Substituting $y = x + \frac{1}{2}$ gives $\frac{1-(2x)^{22}}{1+2x} = \frac{1-(2y-1)^{22}}{2y}$. We want the coefficient of y^2 in the polynomial, so we need to find the coeffienct of y^3 in the numerator and then divide by 2. Using binomial expansion, we get $c_2 = \frac{-2^3 * (-1)^{19} * \binom{22}{3}}{2} = \frac{8*22*21*20}{2*3*2*1} = \boxed{6160}$.

7. Let x, y, and z be positive real numbers with $1 < x < y < z$ such that

$$\log_x y + \log_y z + \log_z x = 8, \text{and}$$
$$\log_x z + \log_z y + \log_y x = \frac{25}{2}.$$

   The value of $\log_y z$ can then be written as $\frac{p+\sqrt{q}}{r}$ for positive integers p, q, and r such that q is not divisible by the square of any prime. Compute $p + q + r$.

   Answer: 42

   Solution: Let $\log_x y = a$, $\log_y z = b$, and $\log_z x = c$, and note that by the Chain Rule, $abc = (\log_x y)(\log_y z)(\log_z x) = \log_x x = 1$. Now, the given system can be written as

$$a + b + c = 8,$$
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{25}{2}.$$

   Expressing the left-hand side of the second equation with a common denominator gives

$$\frac{ab + bc + ac}{abc} = \frac{25}{2}.$$

   Using the fact that $abc = 1$, we obtain the following system of three equations:

$$a + b + c = 8,$$
$$ab + bc + ac = \frac{25}{2},$$
$$abc = 1.$$

   This system is symmetric in a, b, and c and reminiscent of Vieta's formulas. Indeed, a, b, and c are the three roots of the polynomial

$$P(t) = t^3 - 8t^2 + \frac{25}{2}t - 1.$$

   By inspection, we can see that 2 is a root of this polynomial, and factoring out $t-2$ by synthetic division gives

$$P(t) = (t-2)\left(t^2 - 6t + \frac{1}{2}\right).$$

   The second factor has the roots $\frac{6+\sqrt{34}}{2}$ and $\frac{6-\sqrt{34}}{2}$.

   Since $1 < x < y < z$, we also must have $c < a < b$, so that $b = \frac{6+\sqrt{34}}{2}$ and the desired answer is $6 + 34 + 2 = 42$.

8. Find the sum of all possible values of a such that there exists a non-zero complex number z such that the four roots, labeled r_1 through r_4, of the polynomial

$$x^4 - 6ax^3 + (8a^2 + 5a)x^2 - 12a^2x + 4a^2$$

satisfy $|\Re(r_i)| = |r_i - z|$ for $1 \le i \le 4$. Note, for a complex number x, $\Re(x)$ denotes the real component of x.

Answer: 9/17

Solution: The polynomial looks hard to factor. And indeed, if we look at it as a quartic in x, it would be quite difficult. However, by shifting our viewpoint and seeing the polynomial as a quadratic in a, our factorization becomes much more tractable. Our polynomial rewritten in terms of a looks like

$$(4 - 12x + 8x^2)a^2 + (5x^2 - 6x^3)a + x^4.$$

Noticing that the quadratic coefficient can be written as $(1-2x)(4-4x)$ and the linear coefficient can be written as $((1-2x)+(4-4x))x^2$, we can factor our quadratic using elementary techniques. It turns out the factorization is $((1-2x)a + x^2)((4-4x)a + x^2)$. Writing it back in terms of x, we get $(x^2 - 2ax + a)(x^2 - 4ax + 4a)$.

The roots of this polynomial are $a \pm \sqrt{a^2 - a}$ and $2a \pm 2\sqrt{a^2 - a}$. Note that if $a \in (0, 1)$, the roots are all complex. Otherwise, they are all real. In the real case, note that our condition reduces to finding a nonzero complex z such that $|z - r_i| = |r_i|$ for $i \le i \le 4$. Such a z only exists if the four circles defined by the four equations intersect at a nonzero point. However, since all four circles are centered along the real axis and are tangent to the imaginary axis at 0, the only way there exists a nonzero z that is on all four circles is if all 4 circles coincide, which is clearly impossible.

Now consider the complex case when $a \in (0, 1)$. It is not hard to see that z must be real. In that case, for the roots $a \pm \sqrt{a^-a}$, $|\Re(r_i)| = |r_i - z|$ reduces to $a = \sqrt{(a-z)^2 + a - a^2}$, or $z = a \pm \sqrt{2a^2 - a}$. This further restricts a to being greater than $\frac{1}{2}$. Similarly, for the roots $2a \pm 2\sqrt{a^2 - a}$ we get $z = 2a \pm 2\sqrt{2a^2 - a}$. Now the only solution to both of these are if we equate $a + \sqrt{2a^2 - a}$ and $2a - 2\sqrt{2a^2 - a}$. Solving gives $a = 9/17$, our answer.

9. Let $m, n \subset \mathbb{R}$ and

$$f(m, n) = m^4(8 - m^4) + 2m^2n^2(12 - m^2n^2) + n^4(18 - n^4) - 100$$

Find the smallest possible value for a in which $f(m, n) \le a$, regardless of the input of f.

Answer: 69

Solution: Plugging f into the inequality, distributing, and bringing the constants on the right-hand side gives

$$8m^4 - m^8 + 24m^2n^2 - 2m^4n^4 + 18n^4 - n^8 \le a + 100 \quad (1)$$

To simplify things a bit, let $c = a + 100$. Convince yourself, via plugging in $(1, 1)$, that c must be positive. We can refactor the left hand side:

$$2(2m^2 + 3n^2)^2 - (m^4 + n^4)^2 \le c \quad (2)$$

More rearranging yields

$$(2m^2 + 3n^2)^2 \le \frac{c + (m^4 + n^4)^2}{2} \quad (3)$$

The left hand side is nonnegative, and the right hand side is positive. Thus, it must also always be true that

$$2m^2 + 3n^2 \leq \sqrt{\frac{c + (m^4 + n^4)^2}{2}} \tag{4}$$

Again, to simplify things a bit, let $k^2 = c$. The right hand side can be seen as the root mean square (RMS) of k and $m^4 + n^4$. The root mean square of these two is greater than or equal to their geometric mean (GM), $\sqrt{k(m^4 + n^4)}$. But, given (4) and that GM $\leq$ RMS, what does that tell us about the relation between $2m^2 + 3n^2$ and GM? Suppose that k is such that there are some cases where

$$2m^2 + 3n^2 > \sqrt{k(m^4 + n^4)} \tag{5}$$

yet (4) always applies. Now, let any point on the mn plane that satisfies both (5) and

$$m^4 + n^4 = k \tag{6}$$

be denoted as (m_0, n_0). In the case where $m = m_0$ and $n = n_0$, it must be true that

$$2m_0^2 + 3n_0^2 > \sqrt{k(m_0^4 + n_0^4)} = \sqrt{k^2} = k \tag{7}$$

Yet, if (4) still applies, it should also be true that

$$2m_0^2 + 3n_0^2 \leq \sqrt{\frac{k^2 + (m_0^4 + n_0^4)^2}{2}} = \sqrt{\frac{2k^2}{2}} = k \tag{8}$$

This leads to the glaring contradiction that

$$2m_0^2 + 3n_0^2 < 2m_0^2 + 3n_0^2 \tag{9}$$

Any point that satisfies both (5) and (6) cannot satisfy (4). It follows that if (5) were sometimes true for some k, then (4) cannot always be true for that same k. Conversely, if (4) were always true for some k, then (5) can never be true for that same k. Thus, finding the minimum k such that (4) always applies is the same as finding the minimum k such that

$$2m^2 + 3n^2 \leq \sqrt{k(m^4 + n^4)} \tag{10}$$

is always true. Here, we can utilize the Cauchy-Schwarz inequality, where the dot product of two vectors is less than or equal to the product of their magnitudes. Finding a condition for k requires that m^4 and n^4 have the same coefficient - k. The only way to account for that fact is to write $2m^2 + 3n^2$ as the dot product of $\langle 2, 3\rangle$ and $\langle m^2, n^2\rangle$. This results in the inequality

$$2m^2 + 3n^2 \leq \sqrt{(2^2 + 3^2)(m^4 + n^4)} = \sqrt{13(m^4 + n^4)} \tag{11}$$

It follows that $k \geq 13$, $c \geq 169$, and $a \geq 69$. Thus, the smallest possible value for a is $\boxed{69}$.

10. Suppose that the polynomial $x^2 + ax + b$ has the property such that if s is a root, then $s^2 - 6$ is a root. What is the largest possible value of $a + b$?

Answer: 8

Solution: Let $f(s) = s^2 - 6$. Because the roots of are s and $f(s)$, we either have $f(f(s)) = s$ or $f(f(s)) = f(s)$.

We first consider the case $f(f(s)) = f(s)$. Let $r = f(s)$. This gives us $f(r) = r$, or $r^2 - 6 = r$, so $r = -2, 3$. If $r = -2$ and $f(s) = r$, then s must satisfy $s^2 - 6 = -2$, which gives us $s = \pm 2$.

These correspond to the polynomials $x^2 - 4$ and $x^2 + 4x + 4$. On the other hand, if $r = 3$ and $f(s) = r$, then s must satisfy $s^2 - 6 = 3$, which gives us $s = \pm 3$. These correspond to the polynomials $x^2 - 9$ and $x^2 - 6x + 9$. Finally, if $f(s) \neq r$, then we must have $f(s) = s \neq r$, and so we get $r, s = -2, 3$ in some order. This corresponds to the polynomial $x^2 - x - 6$.

We now consider the case $f(f(s)) = s$. Expanding, we get the quartic $s^4 - 12s^2 + 30 = s$, which factors into $(s+2)(s-3)(s^2+s-5) = 0$. Since we have already covered the all the cases where $s = -2, 3$ above, the only new case is when s is a root of $x^2 + x - 5$.

Together, we see that all possible (a, b) are $(0, -4)$, $(4, 4)$, $(0, -9)$, $(-6, 9)$, $(-1, -6)$, and $(1, -5)$. Hence, the maximum value of $a + b$ is given when $(a, b) = (4, 4)$ so $a + b = \boxed{8}$.