1. A standard 6-sided dice is rolled. What is the expected value of the roll given that the value of the result is greater than the expected value of a regular roll?

   Answer: 5

   Solution: The expected value of a single roll is

$$\frac{1}{6} * (1+2+3+4+5+6) = 3.5$$

   Then the condition given means the roll either results in a 4, 5, or 6 with equal probability. The expected value is then

$$\frac{1}{3} * (4+5+6) = \boxed{5}$$

2. 5 students, all with distinct ages, are randomly seated in a row at the movies. The probability that, from left to right, no three consecutive students are seated in increasing age order is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

   Answer: 19

   Solution: From left to right, let A be the event that the first three students are seated in increasing age order, let B be the event that the middle three students are seated in increasing age order, and let C be the event that the last three students are seated in increasing age order.

   Using PIE, $|A|$: Choose 3 students to seat in increasing age order. Seat the last two students in any order. There are $\binom{5}{3} * 2! = 20$ ways to do this. $|B|$ and $|C|$ both follow the same logic, yielding 20 ways each. $|A \cap B|$: this counts the number of ways to seat the first 4 students from left to right in increasing age order. This is $\binom{5}{4} * 1 = 5$. $|B \cap C|$ is identical to $|A \cap B|$, yielding 5 ways. $|A \cap C|$ is the number of ways to seat all 5 students in order, yielding 1 way. $|A \cap B \cap C|$ also the number of ways to seat all 5 students in order, yielding 1 way.

   In total, we get $|A| + |B| + |C|$ - $|A \cap B|$ - $|B \cap C|$ - $|A \cap C| + |A \cap B \cap C| = 20+20+20-5-5-1+1 =$ 50 ways to seat the students in increasing age order.

   Then the number of ways to not seat students in increasing order is $5! - 50 = 120 - 50 = 70$. The probability that no three consecutive students are seated in increasing age order is then $\frac{70}{120} = \frac{7}{12}$. Finally, $7 + 12 = \boxed{19}$.

3. Consider sequences that consist of 0's and 1's. The probability that a random sequence of length 2021 contains an equal number of occurrences of '01' and '10' is $\frac{m}{n}$, where m, n are positive, relatively prime integers. Find $m+n$.

   Answer: 3

   Solution: The substrings 01 and 10 both denote a change in the sequence (from 0 to 1 or vice versa). Therefore, the sequence must have an even number of changes; else, there cannot be an equal number of 01 and 10 substrings. Note that the character 1 can only change to 0 and the character 0 can only change to 1. This means that all sequences that begin and end on the same character have the same number of 01 and 10 substrings, which is exactly $\frac{1}{2}$ the strings. Setting $m = 1$ and $n = 2$, $m + n = 3$.

4. Sofia has a $2 \times 2 \times 2$ wooden cube. She paints each side with a different color and cuts it into 8 unit cubes. Let N be the number of unique ways she can reassemble the unit cubes into a $2 \times 2 \times 2$ cube, given that the painted faces must always be on the outside of the cube. Rotations along any axis of the whole $2 \times 2 \times 2$ cube do <u>not</u> count as distinct.

If N can be written as $a! \cdot p^b$ with p prime, what is $a+b+p$?

Answer: 17.

Solution 1: We note that there are 8 pieces that can be shuffled around in 8! ways, and each rotated 3 ways while keeping the paint on the outside. This yields $8! \cdot 3^8$ permutations, but we need to account for rotations of the whole cube. We mark one of the corners, and divide out 8 for each location it can go in and 3 for each rotation it can be in. This yields

$$N = \frac{8! \cdot 3^8}{8 \cdot 3} = 7! \cdot 3^7$$

and thus $a+b+p = 7+7+3 = 17$.

Solution 2: Alternatively, we keep one corner fixed and permute the other 7. This immediately yields

$$N = 7! \cdot 3^7$$

and the same solution of $a+b+p = 17$.

5. Alice and Bob play a game in which they take turns rolling a fair die. The winner of a round is the first player to roll a 6. Whoever loses rolls first in the next round. Alice rolls first in round 1. The probability that Alice rolls first in round 4 is $\frac{A}{B}$, where A and B are relatively prime, positive integers. Find $A+B$.

Answer: 1996

Solution: In any given round, the first person has a $\frac{1}{6}$ chance of winning on their first roll, a $(\frac{5}{6})^2\frac{1}{6}$ chance of winning on the second roll (since both have to not roll a six), a $(\frac{5}{6})^4\frac{1}{6}$ of winning on the third roll and so on. Overall, their win probability is:

$$\frac{1/6}{1-(5/6)^2} = \frac{6}{11}$$

This means the second person has a win probability of $\frac{5}{11}$. If Alice goes first in round 4 then she must lose round 3. There are four ways to happen based on who wins the first 3 rounds: AAB, ABB, BAB, BBB. The total probability is:

$$\frac{6}{11}\frac{5}{11}\frac{6}{11} + \frac{6}{11}\frac{6}{11}\frac{5}{11} + \frac{5}{11}\frac{6}{11}\frac{6}{11} + \frac{5}{11}\frac{5}{11}\frac{5}{11} = \frac{665}{1331}$$

Therefore, the answer is $\boxed{1996}$.

6. A frog starts at $(0,0)$ and must return to his home at $(13,7)$. There is a river located along the line $y = x-6$. At each step, the frog can only move exactly one unit up or one unit to the right along the lattice points of the plane. If the frog cannot cross the river (but is allowed to move to points on the river), the number of paths the frog can take to his home is N. Compute $\frac{N}{120}$.

Answer: 323

Solution: Disregarding the condition that the frog cannot cross the river, the number of paths to his home is $\binom{20}{7}$ since he must take 20 steps in total and 7 are in the up direction. Now we consider the number of bad paths that cross the river. Any bad path will touch the line $y = x-7$. If we reflect the part of the path after the first point at which it touches this line about $y = x-7$, then the path instead ends at $(14,6)$. Note that any path to $(14,6)$ must cross the river. Thus, we can form a bijection between bad paths and paths that go to $(14,6)$. So, the number of bad paths is $\binom{20}{6}$. Then, the number of allowed paths is $\binom{20}{7} - \binom{20}{6} = \frac{20!}{7!13!}(1-\frac{7}{14}) = \frac{1}{2}\binom{20}{7} = \frac{1}{2}\cdot 20\cdot 19\cdot 17\cdot 4\cdot 3$. So, $\frac{N}{120} = 19\cdot 17 = \boxed{323}$.

7. Anne consecutively rolls a 2020-sided dice with faces labeled from 1 to 2020 and keeps track of the running sum of all her previous dice rolls. She stops rolling when her running sum is greater than 2019. Let X and Y be the running sums she is most and least likely to have stopped at, respectively. What is the ratio between the probabilities of stopping at Y to stopping at X?

   Answer: 2021

   Solution: The running sum that Anne is least likely to have stopped is 4039 because she can only reach it if she previously had a sum of 2019 and then rolled a 2020. Conversely, the running sum she is most likely to have stopped at is 2020, because she could previously have had any sum less than 2020.

   Let $P(S = i)$ be the probability that Anne's running sum S was once exactly i. In general, for $i \le 2020$, $P(S = i) = (1 + \frac{1}{2020})^{i-1} \cdot \frac{1}{2020}$. Thus, the probability that Anne stops with a running sum of 2020 is $(1 + \frac{1}{2020})^{2019} \cdot \frac{1}{2020}$. Anne stops with a running sum of 4039 only if her running sum was previously 2019 and she rolls a 2020. This happens with probability $(1+\frac{1}{2020})^{2018} \cdot \frac{1}{2020}^2$. The ratio between these two probabilities is 2021.

8. Oliver and Xavier are playing a game on an n by n grid of squares. Initially, all cells of the grid are unoccupied. A *turn* is defined as Oliver placing an O on any currently unoccupied square then Xavier placing an X on any remaining unoccupied square. If, at the end of such a turn, there exists a row or column where there are at least 3 more O's than X's in that given row or column, then Oliver wins. Otherwise, if they fill the board and there does not exist such a row or column, Xavier wins. Find the minimum value of n such that Oliver is guaranteed to win if they both play optimally.

   Answer: 5

   Solution: It is easy to see that this is impossible for $n = 1, 2, 3, 4$. To show it is possible for $n = 5$, we note that if after Oliver places an O, there exists a row or column that has two or more O's than X's, then Xavier must place it in that row or column.

   Note that, after four turns, it is always possible for Oliver to place a 2 by 2 square of O's with one square at the center of the 5 by 5 square if they both play optimally Now, since there will only be 4 X's on the board, there will be at least one unoccupied column, call it c. Let a and b be the two rows that the 2 by 2 grid of O's occupies. If they play optimally, at the end of 4 turns, there should be at most 1 X in each of the rows a and b.

   Now, Oliver can place an O in row a, column c. Then, Xavier is forced to play in row a, since now there are two more X's than O's in that row. Next, Oliver can place an O in row b, column c. Similarly, Xavier is forced to play in row b, since now there are two more X's than O's in that row. However, at the end of 6 turns, Oliver has two O's in column c and Xavier has no X's. Therefore, Oliver can always win.

9. Alice plays the violin and piano, and would like to create a practice schedule. She will only practice one instrument on a given day, she can have break days when she does not practice any instrument, and she wants to make sure she does not neglect any instrument for more than two days (e.g. if she does not practice piano for two days, she must practice piano the next day). Following these rules, how many ways can she schedule her practice for eight days?

   Answer: 272

   Solution: Note that the choices of practicing Alice has on a given day depends only on the previous two days. We can split the possible ways she has of practicing during two days into four cases and denote with a_n, b_n, c_n, d_n the number of possible practice schedules for n days ending in those two days: break day and violin/piano (a_n), violin/piano and break day (b_n), violin/piano and piano/violin (alternating instruments, c_n), violin/piano and violin/piano (same instrument for two days, d_n).

From the rules, we see that $a_{n+1} = b_n$, $b_{n+1} = c_n$, $c_{n+1} = a_n + c_n + d_n$, and $d_{n+1} = c_n$. Then, we have

$$a_{n+1} + b_{n+1} + c_{n+1} + d_{n+1} = a_n + b_n + 3c_n + d_n$$
$$= a_n + b_n + c_n + d_n + 2a_{n-1} + 2c_{n-1} + 2d_{n-1}$$
$$= a_n + b_n + c_n + d_n + a_{n-1} + c_{n-1} + d_{n-1} + b_{n-2} + a_{n-2} + c_{n-2} + d_{n-2} + c_{n-2}$$
$$= a_n + b_n + c_n + d_n + a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1} + a_{n-2} + b_{n-2} + c_{n-2} + d_{n-2}.$$

So, the number of possible practice schedules for n days is equal to the sum of the number of possibilities for $n-1, n-2$, and $n-3$ days for $n \geq 5$. We can quickly find that the number of possibilities for 2, 3, and 4 days are 8, 12, and 24. Using the recursion, we get 44 for 5 days, 80 for 6 days, 148 for 7 days, and finally $\boxed{272}$ for 8 days.

10. The chromatic musical scale repeats in groups of 12 pitches: C, $C\sharp$, D, $D\sharp$, E, F, $F\sharp$, G, $G\sharp$, A, $A\sharp$, and B (after B, the next note is C). Define a *chord* as a set of two or more distinct pitches. A *transposition* is a translation of the chord's pitches. Two chords are considered equivalent if they can be obtained from one another through a transposition. For instance, the chords $\{C, E, G, A\sharp\}$ and $\{C\sharp, D\sharp, G, A\sharp\}$ are equivalent because the second chord can be obtained from the first through a transposition of three steps ($C \to D\sharp$, $E \to G$, $G \to A\sharp$, $A\sharp \to C\sharp$, and the order of the notes does not matter). Find the number of distinct chords that can be formed from the set of all twelve pitches.

Answer: 350

Solution: Denote by X the set $\{C, C\sharp, D, D\sharp, E, F, F\sharp, G, G\sharp, A, A\sharp, B\}$. For convenience, let's write it as $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Note that a chord is simply a non-empty subset of X with at least two elements, and we can write it as the 12-tuple $(p_1, p_2, \ldots, p_{12})$, where $p_k = 1$ if pitch k is in the chord and 0 otherwise. Now, let the group $G = \mathbb{Z}/n\mathbb{Z} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ act on X by the following operation: for any $g \in G$, and $x = (p_1, p_2, \ldots, p_{12}) \in X$,

$$g \cdot x = g \cdot (p_1, p_2, \ldots, p_{12}) = (p_{1+g}, p_{2+g}, \ldots, p_{12+g}),$$

where each index is computed modulo 12. We wish to find the number of distinct orbits under this group action.

For ease of computation, consider the empty set and the set with exactly one element chords as well – we'll subtract them from the final count. Then there are 2^{12} chords in all. The identity $e = 0$ fixes all 2^{12} of them.

It is fairly easy to derive the fact that all positive integers relatively prime to 12 fix only the two elements $\{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$ and $\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$. These are 1, 5, 7, and 11 in the set G.

Now, let g be an integer such that $\gcd(g, 12) = d > 1$. We claim that g fixes exactly 2^d elements in X. Let $g = md$ and $12 = nd$, with $\gcd(m, n) = 1$, and let $x = (p_1, p_2, \ldots, p_{12})$ be an element fixed by g. We have

$$\begin{aligned}(p_1, p_2, \ldots, p_{12}) &= g \cdot (p_1, p_2, \ldots, p_{12}) \\ &= \overbrace{(g * g * \cdots * g)}^{\text{k times}} \cdot (p_1, p_2, \ldots, p_{12}) \\ &= kg \cdot (p_1, p_2, \ldots p_{12}) \\ &= (p_{1+kg}, p_{2+kg}, \ldots, p_{12+kg}).\end{aligned}$$

Note that kg is always divisible by d, and since m and n are relatively prime, km can take on all residues $\mod n$. Thus, kg can take on any value from d to $(n-1)d$, inclusive, and taking $k = n$ brings every index i back to itself. So we have $p_i = p_{i+d} = p_{i+2d} = \cdots = p_{i+(n-1)d}$, and after making independent choices for $p_1, p_2, \ldots, p_d$, the rest of the values in the tuple are fixed. Hence, each element g fixes $2^{\gcd(g,12)}$ elements.

That means that $g = 2, 3, 4, 6, 8, 9, 10$ fix $2^2, 2^3, 2^4, 2^6, 2^4, 2^3$, and 2^2 elements, respectively. Plugging these along with the fixed points of e and all elements in g relatively prime to 12 into Burnside's Lemma gives

$$\begin{aligned}k &= \frac{1}{12}\left(2^{12} + 2 + 2^2 + 2^3 + 2^4 + 2 + 2^6 + 2 + 2^4 + 2^3 + 2^2 + 2\right) \\ &= \frac{4096 + 2 + 4 + 8 + 16 + 2 + 64 + 2 + 16 + 8 + 4 + 2}{12} \\ &= \frac{4224}{12} \\ &= 352.\end{aligned}$$

Since this counts the empty set and the set with exactly one element, we subtract 2 to get $\boxed{350}$ distinct chords.