**Time limit:** 50 minutes.
**Instructions:** For this test, you work in teams to solve 15 short answer questions. All answers must be expressed in simplest form unless specified otherwise.
**No calculators.**

1. Square $ABCD$ has side length 2. Let the midpoint of $BC$ be $E$. What is the area of the overlapping region between the circle centered at $E$ with radius 1 and the circle centered at $D$ with radius 2? (You may express your answer using inverse trigonometry functions of non-common values.)

2. Find the number of times $f(x) = 2$ occurs when $0 \le x \le 2022\pi$ for the function
$$f(x) = 2^x(\cos(x) + 1).$$

3. Stanford is building a new dorm for students, and they are looking to offer 2 room configurations:
   - Configuration A: a one-room double, which is a square with side length of $x$;
   - Configuration B: a two-room double, which is two connected rooms, each of them squares with a side length of $y$.

   To make things fair for everyone, Stanford wants a one-room double (rooms of configuration A) to be exactly $1\text{m}^2$ larger than the total area of a two-room double. Find the number of possible pairs of side lengths $(x, y)$, where $x \in \mathbb{N}, y \in \mathbb{N}$, such that $x - y < 2022$.

4. The island nation of Ur is comprised of 6 islands. One day, people decide to create island-states as follows. Each island randomly chooses one of the other five islands and builds a bridge between the two islands (it is possible for two bridges to be built between islands $A$ and $B$ if each island chooses the other). Then, all islands connected by bridges together form an island-state. What is the expected number of island-states Ur is divided into?

5. Let $a, b$, and $c$ be the roots of the polynomial $x^3 - 3x^2 - 4x + 5$. Compute
$$\frac{a^4 + b^4}{a + b} + \frac{b^4 + c^4}{b + c} + \frac{c^4 + a^4}{c + a}.$$

6. Carol writes a program that finds all paths on an 10 by 2 grid from cell $(1, 1)$ to cell $(10, 2)$ subject to the conditions that a path does not visit any cell more than once and at each step the path can go up, down, left, or right from the current cell, excluding moves that would make the path leave the grid. What is the total length of all such paths? (The length of a path is the number of cells it passes through, including the starting and ending cells.)

7. Consider the sequence of integers $a_n$ defined by $a_1 = 1$, $a_p = p$ for prime $p$ and
$$a_{mn} = ma_n + na_m$$
for $m, n > 1$. Find the smallest $n$ such that $\dfrac{a_{n^2}}{2022}$ is a perfect power of 3.

8. Let $\triangle ABC$ be a triangle whose $A$-excircle, $B$-excircle, and $C$-excircle have radii $R_A, R_B$, and $R_C$, respectively (the $A$-excircle is the circle outside $\triangle ABC$ that is tangent to $BC, \overrightarrow{AB}$, and $\overrightarrow{AC}$—the other excircles are defined similarly). If $R_A R_B R_C = 384$ and the perimeter of $\triangle ABC$ is 32, what is the area of $\triangle ABC$?

9. Consider the set $S$ of functions $f : \{1, 2, \ldots, 16\} \to \{1, 2, \ldots, 243\}$ satisfying:

   (a) $f(1) = 1$

   (b) $f(n^2) = n^2 f(n)$,

   (c) $n \mid f(n)$,

   (d) $f(\mathrm{lcm}(m, n))f(\gcd(m, n)) = f(m)f(n)$.

   If $|S|$ can be written as $p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_k^{e_k}$ where $p_i$ are distinct primes, compute $p_1e_1+p_2e_2+\ldots+p_ke_k$.

10. You are given that $\log_{10} 2 \approx 0.3010$ and that the first (leftmost) two digits of $2^{1000}$ are 10. Compute the number of integers $n$ with $1000 \le n \le 2000$ such that $2^n$ starts with either the digit 8 or 9 (in base 10).

11. Let $O$ be the circumcenter of $\triangle ABC$. Let $M$ be the midpoint of $BC$, and let $E$ and $F$ be the feet of the altitudes from $B$ and $C$, respectively, onto the opposite sides. $EF$ intersects $BC$ at $P$. The line passing through $O$ and perpendicular to $BC$ intersects the circumcircle of $\triangle ABC$ at $L$ (on the major arc $BC$) and $N$, and intersects $BC$ at $M$. Point $Q$ lies on the line $LA$ such that $OQ$ is perpendicular to $AP$. Given that $\angle BAC = 60^\circ$ and $\angle AMC = 60^\circ$, compute $OQ/AP$.

12. Let $\mathcal{T}$ be the isosceles triangle with side lengths $5, 5, 6$. Arpit and Katherine simultaneously choose points $A$ and $K$ within this triangle, and compute $d(A, K)$, the squared distance between the two points. Suppose that Arpit chooses a random point $A$ within $\mathcal{T}$. Katherine plays the (possibly randomized) strategy which given Arpit's strategy minimizes the expected value of $d(A, K)$. Compute this value.

13. For a regular polygon $S$ with $n$ sides, let $f(S)$ denote the regular polygon with $2n$ sides such that the vertices of $S$ are the midpoints of every other side of $f(S)$. Let $f^{(k)}(S)$ denote the polygon that results after applying $f$ a total of $k$ times. The area of

$$\lim_{k\to\infty} f^{(k)}(P)$$

where $P$ is a pentagon of side length 1, can be expressed as $\frac{a+b\sqrt{c}}{d}\pi^m$ for some positive integers $a, b, c, d, m$ where $d$ is not divisible by the square of any prime and $d$ does not share any positive divisors with $a$ and $b$. Find $a + b + c + d + m$.

14. Consider the function

$$f(m) = \sum_{n=0}^{\infty} \frac{(n-m)^2}{(2n)!}.$$

This function can be expressed in the form $f(m) = \frac{a_m}{e} + \frac{b_m}{4}e$ for sequences of integers $\{a_m\}_{m\ge1}, \{b_m\}_{m\ge1}$. Determine

$$\lim_{m\to\infty} \frac{2022b_m}{a_m}.$$

15. In $\triangle ABC$, let $G$ be the centroid and let the circumcenters of $\triangle BCG, \triangle CAG$, and $\triangle ABG$ be $I, J$, and $K$, respectively. The line passing through $I$ and the midpoint of $BC$ intersects $KJ$ at $Y$. If the radius of circle $K$ is 5, the radius of circle $J$ is 8, and $AG = 6$, what is the length of $KY$?