1. Square $ABCD$ has side length 2. Let the midpoint of BC be E. What is the area of the overlapping region between the circle centered at E with radius 1 and the circle centered at D with radius 2? (You may express your answer using inverse trigonometry functions of non-common values.)

   Answer: $\mathbf{3\arctan\left(\frac{1}{2}\right)+\frac{\pi}{2}-2}$

   Solution: We can find the overlapping area by adding together the sector of circle D along minor arc CF and the sector of circle E along minor arc CF, and subtracting $CDFE$. The area of $CDFE$ is 2 (formed by adding together $\triangle CDE$ and $\triangle FDE$, each with area 1). Then, the area of the sector of circle D is $\frac{2\arcsin\left(\frac{1}{\sqrt{5}}\right)}{2\pi}\cdot 4\pi$ and the area of the sector of circle E is $\frac{2\arcsin\left(\frac{2}{\sqrt{5}}\right)}{2\pi}\cdot\pi$. Our answer then is

$$\begin{aligned}\frac{2\arcsin\left(\frac{1}{\sqrt{5}}\right)}{2\pi}\cdot 4\pi+\frac{2\arcsin\left(\frac{2}{\sqrt{5}}\right)}{2\pi}\cdot\pi-2 &= 4\arcsin\left(\frac{1}{\sqrt{5}}\right)+\arcsin\left(\frac{2}{\sqrt{5}}\right)-2\\ &= 4\arcsin\left(\frac{1}{\sqrt{5}}\right)+\frac{\pi}{2}-\arcsin\left(\frac{1}{\sqrt{5}}\right)-2\\ &= 3\arcsin\left(\frac{1}{\sqrt{5}}\right)+\frac{\pi}{2}-2\\ &= \boxed{3\arctan\left(\frac{1}{2}\right)+\frac{\pi}{2}-2}.\end{aligned}$$

   Note that there are other equivalent forms of the answer.

2. Find the number of times $f(x)=2$ occurs when $0\le x\le 2022\pi$ for the function

$$f(x)=2^x(\cos(x)+1).$$

   Answer: 2023

   Solution: In the following n always denotes an integer between 0 and 2022.
   We know that $\cos(x)+1=2$ for $x=2n\pi$, and that $\cos(x)+1=0$ for $x=(2n+1)\pi$.
   By the intermediate value theorem, it is obvious that in the interval $n\pi<x<(n+1)\pi$, $\cos(x)+1$ attains all values between 0 and 2 exclusive.
   We have that for all intervals $n\pi<x<(n+1)\pi$ in the domain that $0<2^{-x}<2$, and so we must have some $0<x<(n+1)\pi$ such that $\cos(x)+1=2^{-x}$.
   This finds us 2022 solutions, since there are 2022 such disjoint open intervals of the form $(n\pi, (n+1)\pi)$ in the domain.
   Then all that remain are to consider the points where $x=n\pi$ (the endpoints of these intervals) which we easily see can only leave the additional solution $x=0$. This finds us our additional one solution, so there are $2022+1=\boxed{2023}$ solutions in total.

3. Stanford is building a new dorm for students, and they are looking to offer 2 room configurations:

   - Configuration A: a one-room double, which is a square with side length of x;
   - Configuration B: a two-room double, which is two connected rooms, each of them squares with a side length of y.

To make things fair for everyone, Stanford wants a one-room double (rooms of configuration A) to be exactly 1m^2 larger than the total area of a two-room double. Find the number of possible pairs of side lengths (x, y), where $x \in \mathbb{N}, y \in \mathbb{N}$, such that $x - y < 2022$.

Answer: 5

Solution: We are looking for solutions to

$$x^2 - 2y^2 = 1.$$

Note that this is an example of Pell's equation, and so if (x_0, y_0) is the smallest possible solution (also called fundamental solution), we can generate an infinite array of solutions via the recurrent formulas

$$x_{n+1} = x_0 x_n + 2y_0 y_n$$
$$y_{n+1} = y_0 x_n + x_0 y_n.$$

It is easy to observe that (3, 2) is the fundamental solution, and from there we can easily build the solutions (3, 2), (17, 12), (17, 12), (99, 50), (497, 348), (2883, 2038), and the next one is (16801, 11880), which is clearly has a difference larger than 2022 (note than the question only asks for the number of solutions, so these need not be precisely calculated). Therefore, the answer is 5.

4. The island nation of Ur is comprised of 6 islands. One day, people decide to create island-states as follows. Each island randomly chooses one of the other five islands and builds a bridge between the two islands (it is possible for two bridges to be built between islands A and B if each island chooses the other). Then, all islands connected by bridges together form an island-state. What is the expected number of island-states Ur is divided into?

Answer: $\mathbf{\frac{3493}{3125}}$

Solution 1: Consider the directed graph on 6 vertices formed by considering each island as a vertex and each bridge as a directed edge from the island which constructed the bridge. Then, we see that each vertex has out-degree 1 and the expected number of island-states or connected components is exactly the number of cycles formed in this graph. For $i = 2, 3, \ldots, 6$, there are $\binom{6}{i}(i-1)!$ difference cycles of length i and the probability of each occurring is $\frac{1}{5^i}$, thus the expected number of cycles and thus island states is

$$\sum_{i=2}^{6} \frac{\binom{6}{i}(i-1)!}{5^i} = \boxed{\frac{3493}{3125}}.$$

Solution 2: There are only a few ways that the islands can be divided into island-states, since it is not possible for one island to form an island-state by itself. The possibilities are $2+2+2$, $2+4$, $3+3$, and 6.

Case 2 + 2 + 2: We can split the island-states in $\frac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{3!} = 15$ ways.

Case 2 + 4: We can split the island-states in $\binom{6}{2} = 15$ ways. Within the island-state of size 4, each island can choose to build a bridge to any other island, except we cannot allow them to be partitioned as $2+2$. There are 3 ways to partition them as $2+2$ (island a can be paired with b, c, or d), so we subtract to get $3^4 - 3 = 78$. Then, the total number of ways to build the bridges for this case is $15 \cdot 78 = 1170$.

Case 3 + 3: We can split the island-states in $\frac{\binom{6}{3}\binom{3}{3}}{2!} = 10$ ways. Within each island-state of size

3, it is not possible for the island-state to be partitioned into smaller island-states, so each island can connect to any other island, giving us $10 \cdot 2^3 \cdot 2^3 = 640$.
Case 6: We subtract the number of possibilities for the previous case from the total number of possibilities to get $5^6 - 640 - 1170 - 15 = 13800$.
Thus, the expected number of island-states is $\frac{15 \cdot 3 + 1170 \cdot 2 + 640 \cdot 2 + 13800 \cdot 1}{5^6} = \boxed{\frac{3493}{3125}}$.

5. Let a, b, and c be the roots of the polynomial $x^3 - 3x^2 - 4x + 5$. Compute

$$\frac{a^4+b^4}{a+b} + \frac{b^4+c^4}{b+c} + \frac{c^4+a^4}{c+a}.$$

Answer: $\mathbf{\frac{869}{7}}$

Solution: Vieta's formulas tell us that $a + b + c = 3$, and thus we can rewrite the expression has

$$\frac{a^4+b^4}{3-c} + \frac{b^4+c^4}{3-a} + \frac{c^4+a^4}{3-b}$$

Combining everything under a common denominator gives us

$$\frac{(a^4+b^4)(3-a)(3-b) + (b^4+c^4)(3-b)(3-c) + (c^4+a^4)(3-c)(3-a)}{(3-a)(3-b)(3-c)}$$

Denote $S_k = a^k + b^k + c^k$. Now, by expanding the numerator and regrouping terms, we obtain

$$\frac{18S_4 - 6S_5 - 3S_4S_1 + 3S_5 + S_5S_1 - S_6}{27 - 9(a+b+c) + 3(ab+bc+ca) - abc}$$

We can derive from Vieta's formulas and Newton's sums that $S_1 = 3$, $S_2 = 17$, and $S_3 = 48$. Now, using the fact that $S_k = 3S_{k-1} + 4S_{k-2} - 5S_{k-3}$, then we can compute $S_3 = 48$, $S_4 = 197$, $S_5 = 698$, and $S_6 = 2642$. Putting this all together, we get a value of $\boxed{\frac{869}{7}}$.

6. Carol writes a program that finds all paths on an 10 by 2 grid from cell $(1, 1)$ to cell $(10, 2)$ subject to the conditions that a path does not visit any cell more than once and at each step the path can go up, down, left, or right from the current cell, excluding moves that would make the path leave the grid. What is the total length of all such paths? (The length of a path is the number of cells it passes through, including the starting and ending cells.)

Answer: 7680

Solution: First, we claim that the number of such paths for a n by 2 grid, which we denote t_n, is 2^{n-1}. The number of such paths from $(1, 1)$ to $(n, 1)$, which we denote b_n, is also 2^{n-1}. We can prove this by induction. Note that any such path cannot move to the right without visiting a previous cell again. For our base case, we see that in a 1 by 2 grid, clearly there is only 1 path to $(1, 2)$ and only 1 path to $(1, 1)$ (the path that just consists of the cell $(1, 1)$). For the inductive step, we see that the number of paths to cell $(n, 2)$ is equal to the sum of the number of paths to $(n - 1, 1)$ and $(n - 1, 2)$, and we have the same for $(n, 1)$. This means that $t_n = t_{n-1} + b_{n-1} = 2^{n-2} + 2^{n-2} = 2^{n-1}$. Similarly, we also have $b_n = 2^{n-1}$, completing the inductive step.
Now, to count the total length of all paths, we can count the number of paths passing through each cell and add all of these values. For a cell $(k, 2)$ where $1 < k < n$, we see that the number

of paths passing through is $t_{k-1}b_{n-k+1} + b_{k-1}b_{n-k} = 2^{k-2}2^{n-k} + 2^{k-2}2^{n-k-1} = 3\left(2^{n-3}\right)$. We can see similarly that the number of paths passing through $(k, 1)$ is $(3)\left(2^{n-3}\right)$.

Now we look at the cells $(1, 1)$, $(1, 2)$, $(n, 1)$, and $(n, 2)$. Every path passes through the starting and ending cells, so 2^{n-1} paths each pass through $(1, 1)$ and $(n, 2)$. If a path goes to cell $(1, 2)$, then the number of paths to $(n, 2)$ is $b_{n-1} = 2^{n-2}$. Similarly, the number of paths through $(n, 1)$ is also 2^{n-2}. Summing these values, we have $2^{n-1} + 2^{n-1} + 2^{n-2} + 2^{n-2} = (3)\left(2^{n-1}\right)$. Our total then is $(3)\left(2^{n-1}\right) + (2n-4)(3)\left(2^{n-3}\right) = (3)\left(2^{n-1} + (n-2)\left(2^{n-2}\right)\right)$. Letting $n = 10$ gives $(3)(2^9 + (8)(2^8)) = \boxed{7680}$.

7. Consider the sequence of integers a_n defined by $a_1 = 1$, $a_p = p$ for prime p and

$$a_{mn} = ma_n + na_m$$

for $m, n > 1$. Find the smallest n such that $\dfrac{a_{n^2}}{2022}$ is a perfect power of 3.

Answer: 3^{337}

Solution 1: Dividing by mn on both sides of $a_{mn} = ma_n + na_m$ we get

$$\frac{a_{mn}}{mn} = \frac{a_n}{n} + \frac{a_m}{m}.$$

Therefore, we can build up to

$$\frac{a_{n_1 n_2 \ldots n_k}}{n_1 n_2 \ldots n_k} = \sum_{i=1}^{k} \frac{a_{n_i}}{n_i}$$

Now, if the canonical prime factorization of n is $n = \prod_i p_i^{\alpha_i}$, we get

$$n^2 = \prod_i p_i^{2\alpha_i} \implies \frac{a_{n^2}}{n^2} = \sum_i \frac{2\alpha_i a_{p_i}}{p_i} = 2\sum_i \frac{\alpha_i}{p_i}$$

Multiplying by n^2 on both sides, we can get an expression for a_{n^2} as

$$a_{n^2} = 2\left(\prod_i p_i^{2\alpha_i}\right)\sum_i \frac{\alpha_i a_{p_i}}{p_i}$$

We want $\frac{a_{n^2}}{2022}$ to be a perfect power of 3, hence $a_{n^2} = 2022 \cdot 3^r = 2 \cdot 3^{r+1} \cdot 337$. Since 2 is already a factor in a_{n^2}, we can conclude that n can have at most 2 prime factors, namely 3 and 337.

Case 1: $n = p^\alpha \Leftrightarrow \alpha p^{2\alpha-1} = 337 \cdot 3^{r+1}$. Obviously, $p \neq 337$, since $r > 0$, and so, $p = 3, \alpha = 337$, and we get $n = 3^{337}$.

Case 2: $n = 3^\alpha \cdot 337$. It is easy to check that 337 cannot be raised to any power beyond 1. Now we have

$$a_{n^2} = 2 \cdot 3^{2\alpha-1} \cdot 337(337\alpha + 3) = 2 \cdot 3^{r+1} \cdot 337$$

Therefore, we have $337\alpha + 3 = 3^t$ for some t. From here, we can express $\alpha = \frac{3^t-3}{337}$, from which follows that $3^{t-1} \equiv 1 (\texttt{mod}\ 337)$. Note that we are looking only for candidate values of $\alpha < 337$, or equivalently $5 < t < 11$. We can quickly check that none of the values for t work (note that the smallest t is 169, i.e. the multiplicative order $\texttt{ord}_{337}(3) = 168$), and so, the minimum solution is $n = 3^{337}$.

Solution 2: Begin as before, having $\frac{a_{mn}}{mn} = \frac{a_m}{m} + \frac{a_n}{n}$. Now, let $b_n = \frac{a_n}{n}$. Then, if $n = p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_k^{e_k}$, we can see that $b_n = \sum_{i=1}^{k} e_k$ by removing one such factor and inducting.

In addition, $b_{n^2} = 2b_n$ so $a_{n^2} = 2n^2 b_n$. Hence, $\frac{a_{n^2}}{2\cdot 3\cdot 337} = \frac{n^2\cdot b_n}{3\cdot 337}$. This implies that $337 \nmid n$, as otherwise $\frac{n^2}{337}$ would have an extra factor. Hence, $n^2 = 3^{2k}$ for some k, and $n^2 \cdot b_n = k \cdot 3^{2k}$. Therefore, we need $k = 337$ (since $337 \mid k$) for minimality and therefore $n = 3^{337}$.

8. Let $\triangle ABC$ be a triangle whose A-excircle, B-excircle, and C-excircle have radii R_A, R_B, and R_C, respectively (the A-excircle is the circle outside $\triangle ABC$ that is tangent to $BC, \overrightarrow{AB}$, and $\overrightarrow{AC}$—the other excircles are defined similarly). If $R_A R_B R_C = 384$ and the perimeter of $\triangle ABC$ is 32, what is the area of $\triangle ABC$?

Answer: 24

Solution: Let the A-excircle be tangent to $BC, \overrightarrow{AB}$, and $\overrightarrow{AC}$ at D, E, and F, respectively, and let the center of the A-excircle be O_A. Since the A-excircle is tangent at these points, we have $BD = BE$ and $CD = CF$. Then,

$$\begin{aligned}[ABC] &= [AEO_A] + [AFO_A] - [BDO_AE] - [CDO_AF] \\ &= \frac{1}{2}R_A(AB + BE + AC + CF - BD - BE - CD - CF) \\ &= \frac{1}{2}R_A(AB + AC - BC) \\ &= R_A(s - BC),\end{aligned}$$

where s is the semiperimeter of $\triangle ABC$. So, $R_A = \frac{[ABC]}{s-BC}$ and we can also see that $R_B = \frac{[ABC]}{s-AC}$, $R_C = \frac{[ABC]}{s-AB}$. Then,

$$\begin{aligned}R_A R_B R_C &= \frac{[ABC]^3}{(s-BC)(s-AC)(s-AB)} \\ &= \frac{s[ABC]^3}{s(s-BC)(s-AC)(s-AB)} \\ &= \frac{s[ABC]^3}{[ABC]^2} \\ &= s[ABC].\end{aligned}$$

Finally, we have $[ABC] = \frac{R_A R_B R_C}{s} = \frac{384}{16} = \boxed{24}$.

9. Consider the set S of functions $f : \{1, 2, \ldots, 16\} \to \{1, 2, \ldots, 243\}$ satisfying:

   (a) $f(1) = 1$
   (b) $f(n^2) = n^2 f(n)$,
   (c) $n \mid f(n)$,
   (d) $f(\text{lcm}(m, n))f(\gcd(m, n)) = f(m)f(n)$.

If $|S|$ can be written as $p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_k^{e_k}$ where p_i are distinct primes, compute $p_1e_1 + p_2e_2 + \ldots + p_ke_k$.

Answer: 91

Solution: Let $g(n) = \frac{f(n)}{n}$. Then, $g(n^2) = ng(n)$ and the fourth property is still satisfied for g. The only major change is the range on g.

Note that the first property guarantees that $g(1) = 1$. In addition, if $\gcd(m, n) = 1$, then we have $f(mn) = f(m)f(n)$, so f is multiplicative. So, it suffices to specify $g(p^k)$. By the first property, given $g(p)$ we have all $g(p^{2^i})$. Therefore, we only need to give $g(p^k)$ for odd k.

Notice that if $p^k \geq 8$, then $g(p^k)$ can take on any value from 1 to $243/p^k$ (as we can never multiply by it and still be within range of 16). The only three possibilities are 8, 11, and 13. In total, these contribute $30 \cdot 22 \cdot 18$. Now, we must have $g(16) = 8g(2) \leq m/16$, so $g(2) \leq \frac{243}{128}$, so $g(2) = 1$. In addition, we must have $g(9) = 3g(3) \leq m/9$, so $g(3) \leq \frac{243}{27} = 9$.

So, the remaining unspecified values are $g(5)$ and $g(7)$. We can bound these via $g(15) = g(3)g(5) \leq \frac{243}{15} = 16$ and $g(14) = g(2)g(7) = g(7) \leq \frac{243}{14} = 17$.

So, the total number of solutions is $30 \cdot 22 \cdot 18 \cdot 17 \cdot (16+8+5+4+3+2+2+2+1) = 2 \cdot 3 \cdot 5 \cdot 2 \cdot 11 \cdot 2 \cdot 3^2 \cdot 17 \cdot 43 = 2^3 \cdot 3^3 \cdot 5 \cdot 11 \cdot 17 \cdot 43$. The answer then is $6+9+5+11+17+43 = \boxed{91}$.

10. You are given that $\log_{10} 2 \approx 0.3010$ and that the first (leftmost) two digits of 2^{1000} are 10. Compute the number of integers n with $1000 \leq n \leq 2000$ such that 2^n starts with either the digit 8 or 9 (in base 10).

Answer: 97

Solution: Consider the "runs" of first digits that can happen. These are $(1, 2, 4, 8)$; $(1, 2, 4, 9)$; $(1, 2, 5)$; $(1, 3, 6)$; $(1, 3, 7)$. So, 2^n starts with the digit 8 or 9 if and only if it has a length 4 run until we get another decimal digit.

So, if let x be the number of runs of length 3 and y be the number of runs of length 4. If 2^n starts with a 1, then that means $2^0, 2^1, \ldots, 2^{n-1}$ have $3x + 4y = n$ total terms, and since each run ends with getting a new digit, $x + y = \lfloor \log_{10} 2^{n-1} + 1 \rfloor$. So, $y = n - 3\lfloor (n-1)\log_{10} 2 + 1 \rfloor$.

Since 2^{1000} starts with a 1, the number of 2^k starting with 8 or 9 with $k \leq 1000$ is $1000 - 3\lfloor 999 \log_{10} 2 + 1 \rfloor = 97$.

Since 2^{1000} starts with 10, then 2^{2000} starts with 1 as well.

Therefore, the number of 2^k starting with 8 or 9 with $k \leq 2000$ is $2000 - 3\lfloor 1999 \log_{10} 2 + 1 \rfloor = 194$. Therefore, our answer is $194 - 97 = \boxed{97}$.

11. Let O be the circumcenter of $\triangle ABC$. Let M be the midpoint of BC, and let E and F be the feet of the altitudes from B and C, respectively, onto the opposite sides. EF intersects BC at P. The line passing through O and perpendicular to BC intersects the circumcircle of $\triangle ABC$ at L (on the major arc BC) and N, and intersects BC at M. Point Q lies on the line LA such that OQ is perpendicular to AP. Given that $\angle BAC = 60°$ and $\angle AMC = 60°$, compute OQ/AP.

Answer: $\mathbf{\frac{2}{\sqrt{3}}}$

Solution: Let H be the orthocenter of $\triangle ABC$ and draw MH. Notice that $MH \parallel OQ$ so

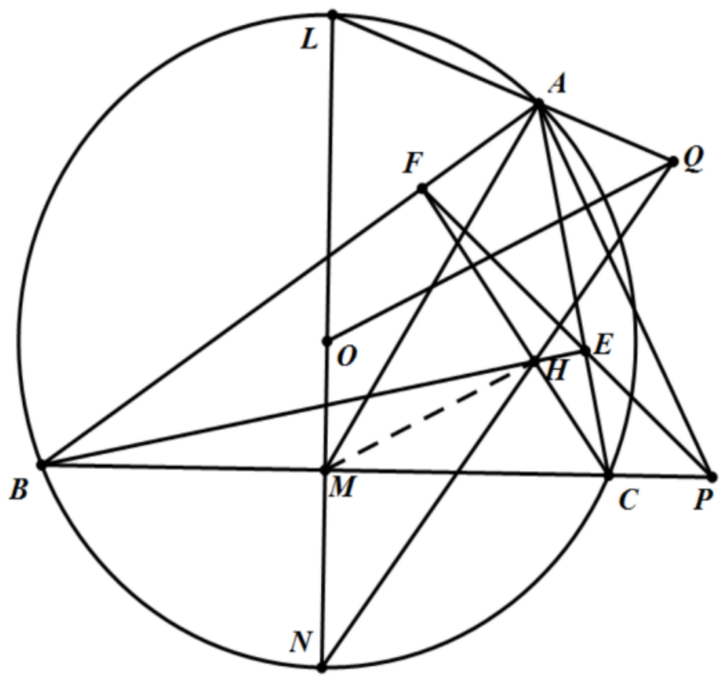
$$OQ = MH \cdot \frac{ON}{MN} = MH \cdot \frac{ON}{NC \sin\left(\frac{1}{2}\angle BAC\right)} = 2MH.$$

H is the orthocenter of $\triangle AMP$ so $AP = \sqrt{3}MH$. $OQ/AP = \boxed{\frac{2}{\sqrt{3}}}$.

To see that $MH \parallel OQ$, note first that $OM \parallel AH$ since they are both perpendicular to BC. Let OQ intersect AH at point O'. Then, it suffices to show that $OM = O'H$. Since $\triangle QLN \sim \triangle QAH$, we see that because O is the midpoint of LN, O' is the midpoint of AH. Then, what we want to show is equivalent to $OM = \frac{1}{2}AH$. Let CO intersect the circumcircle again at C'. Since $BC' = 2OM$, we want to show that $BC' = AH$. We know BCC' is a 30-60-90 triangle, so

$BC' = \frac{1}{2}CC' = R$, where R is the circumradius of $\triangle ABC$. We see that $\angle C'BH = \angle C'BA + \angle ABH = \angle C'CA + 90^\circ - \angle A = \angle C - 30^\circ + 90^\circ - 60^\circ = \angle C$. Also, $\angle BC'A = 180^\circ - \angle C$. This proves that $C'BHA$ is a parallelogram, so $BC' = AH = R$, proving the claim.

Note that we only need the fact that $\angle A = 60^\circ$ to say that $AH = R$. This means in $\triangle AMP$ we have $MH = R'$, where R' is the circumradius of $\triangle AMP$. We find that $AP = \sqrt{3}R'$ from the Law of Sines, which means $AP = \sqrt{3}MH$.



12. Let $\mathcal{T}$ be the isosceles triangle with side lengths $5, 5, 6$. Arpit and Katherine simultaneously choose points A and K within this triangle, and compute $d(A, K)$, the squared distance between the two points. Suppose that Arpit chooses a random point A within $\mathcal{T}$. Katherine plays the (possibly randomized) strategy which given Arpit's strategy minimizes the expected value of $d(A, K)$. Compute this value.

Answer: $\mathbf{\frac{43}{18}}$

Solution: Place the triangle such that the vertices are $A = (-3, 4)$, $B = (0, 0)$, and $C = (3, 4)$. We claim that Katherine's optimal strategy is to deterministically play the centroid of this triangle: $(0, \frac{8}{3})$. Notice that the area of this triangle is 12, so we have a uniform distribution over the triangle with $p(x, y) = \frac{1}{12}$. To parametrize the triangle, notice that for a given y, the range of x is $[-\frac{3}{4}y, \frac{3}{4}y]$. So, the expected value of $d(A, K)$ is

$$\begin{aligned}
\frac{1}{12}\int_0^4 \int_{-\frac{3}{4}y}^{\frac{3}{4}y} \left[x^2 + \left(\frac{8}{3} - y\right)^2\right] \mathrm{d}x\,\mathrm{d}y &= \frac{1}{12}\int_0^4 \left[\frac{x^3}{3} + \left(\frac{8}{3} - y\right)^2 \cdot x\Big|_{x=-\frac{3}{4}y}^{\frac{3}{4}y}\right] \mathrm{d}y \\
&= \frac{1}{12}\int_0^4 \frac{57}{32}y^3 - 8y^2 + \frac{32}{3}y\,\mathrm{d}y \\
&= \frac{1}{12}\left[\frac{57}{128}y^4 - \frac{8}{3}y^3 + \frac{16}{3}y^2\Big|_{y=0}^{4}\right] \\
&= \frac{1}{2}\left(114 - \frac{512}{3} + \frac{256}{3}\right) \\
&= \frac{86}{36} \\
&= \boxed{\frac{43}{18}}.
\end{aligned}$$

Now we show that this is indeed the optimal strategy. Let $p(x, y)$ be Katherine's strategy: in other words, she selects point (x, y) with "probability" $p(x, y)$.

We first note that it is optimal for Katherine to only play along $x = 0$. Indeed, consider any strategy p for Katherine, and let p' be such that $p'(0, y) = \int_{-\frac{3}{4}y}^{\frac{3}{4}y} p(x, y)\,\mathrm{d}x$ and $p'(x, y) = 0$ otherwise. Then consider the following process: Arpit selects a random point (a, b) with $a \geq 0$ and then with probability $\frac{1}{2}$ flips a. Note that this is equivalent to choosing a random point in the triangle. So, he selects one of (a, b) and $(-a, b)$ at random. We consider the contribution of Katherine's strategy to the expected value along $y = b$ (note that we can fix any y, this just allows the y term to be 0). Then, Katherine's strategy incurs loss (proportional to)

$$\frac{1}{2}\left[\int_{-\frac{3}{4}y}^{\frac{3}{4}y} (x - a)^2 p(x, y)\,\mathrm{d}x + \int_{-\frac{3}{4}y}^{\frac{3}{4}y} (x + a)^2 p(x, y)\,\mathrm{d}x\right] = \int_{-\frac{3}{4}y}^{\frac{3}{4}y} [x^2 + a^2] p(x, y)\,\mathrm{d}x \geq a^2 p'(0, y)$$

where equality is reached exactly where $p = p'$. Generalizing over all points Arpit could choose, we still have that p' outperforms p.

So, Katherine's strategy p is only nonzero along the y-axis. So, it suffices for us to only look at the contribution from y. Note that Arpit has y-coordinate c with probability $\frac{\frac{3}{4}c - (-\frac{3}{4}c)}{12} = \frac{c}{8}$. So, suppose Arpit chooses a coordinate c and Katherine chooses a coordinate y. Then, the expected loss is

$$\int_0^4 \int_0^4 (y - c)^2 \cdot \frac{c}{8} \cdot p(0, y)\,\mathrm{d}c\,\mathrm{d}y = \int_0^4 [y^2 - \frac{16}{3}y + 8] p(0, y)\,\mathrm{d}y.$$

So, it suffices to minimize the inner quantity and set $p(0, y) = 1$ at that minimizer y. This is a parabola with vertex $\frac{8}{3}$, so this implies that Katherine's strategy should be to always play $(0, \frac{8}{3})$ and we are done.

13. For a regular polygon S with n sides, let $f(S)$ denote the regular polygon with $2n$ sides such that the vertices of S are the midpoints of every other side of $f(S)$. Let $f^{(k)}(S)$ denote the polygon that results after applying f a total of k times. The area of

$$\lim_{k \to \infty} f^{(k)}(P)$$

where P is a pentagon of side length 1, can be expressed as $\frac{a+b\sqrt{c}}{d}\pi^m$ for some positive integers a, b, c, d, m where d is not divisible by the square of any prime and d does not share any positive divisors with a and b. Find $a+b+c+d+m$.

Answer: 141

Solution: First, note that the centers of all the polygons $P, f(P), f^2(P), \dots$ are identical. Denote this point by O This is because circumscribing a polygon around another in the manner described by f does not change the center. Furthermore, the limiting polygon approaches a circle. Therefore, it suffices to find the radius of this circle.

Let r_n denote distance from O to a vertex of $f^{(n)}(P)$ (where $f^0(P) = P$). Then we know that $r_0 = \frac{1}{2\sin(\pi/5)}$, and we also have the relation

$$r_n = r_{n-1}\frac{1}{\cos\left(\frac{\pi}{5\cdot 2^n}\right)}.$$

Therefore, we have

$$r_n = r_0\prod_{i=1}^{n}\frac{1}{\cos\left(\frac{\pi}{5\cdot 2^i}\right)}.$$

We can condense this by multiplying by $\frac{\sin\left(\frac{\pi}{5\cdot 2^n}\right)}{\sin\left(\frac{\pi}{5\cdot 2^n}\right)}$ and then repeatedly applying the formula $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ to the denominator. This gives us

$$r_n = r_0 2^n\frac{\sin\left(\frac{\pi}{5\cdot 2^n}\right)}{\sin\left(\frac{\pi}{5}\right)}$$

As n approaches infinity, we have $\sin\left(\frac{\pi}{5\cdot 2^n}\right) \approx \frac{\pi}{5\cdot 2^n}$, which means r_n approaches $\frac{\pi/5}{2\sin^2(\pi/5)} = \frac{\pi}{10\sin^2(\pi/5)}$ (we can make this more rigorous with the Taylor expansion of sin, but that's not necessary for now). Thus, the area of the limiting figure is a circle with area

$$\pi\frac{\pi^2}{100\sin^4(\pi/5)} = \frac{6+2\sqrt{5}}{125}\pi^3$$

which gives us an answer of $\boxed{141}$.

14. Consider the function

$$f(m) = \sum_{n=0}^{\infty}\frac{(n-m)^2}{(2n)!}.$$

This function can be expressed in the form $f(m) = \frac{a_m}{e} + \frac{b_m}{4}e$ for sequences of integers $\{a_m\}_{m\geq 1}, \{b_m\}_{m\geq 1}$. Determine

$$\lim_{m\to\infty}\frac{2022b_m}{a_m}.$$

Answer: 8088

Solution: Expanding we have

$$f(m) = \sum_{n=0}^{\infty}\frac{n^2}{(2n)!} - 2m\sum_{n=0}^{\infty}\frac{n}{(2n)!} + m^2\sum_{n=0}^{\infty}\frac{1}{(2n)!}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{n}{(2n-1)!} - m\sum_{n=1}^{\infty}\frac{1}{(2n-1)!} + m^2\sum_{n=0}^{\infty}\frac{1}{(2n)!}$$

Let's look at that last summation first. It is the same as

$$m^2\sum_{k=0}^{\infty}\frac{1}{2}\left[\frac{1}{k!}+\frac{(-1)^k}{k!}\right] = \frac{m^2}{2}\left(e+\frac{1}{e}\right)$$

the second summation is the same as

$$-m\sum_{k=1}^{\infty}\frac{1}{2}\left[\frac{1}{k!}-\frac{(-1)^k}{k!}\right] = -\frac{m}{2}\left(e-\frac{1}{e}\right)$$

Finally, the last summation is the same as

$$\frac{1}{4}\sum_{n=1}^{\infty}\frac{(2n-1)+1}{(2n-1)!} = \frac{1}{4}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-2)!}+\frac{1}{(2n-1)!}\right] = \frac{1}{4}\left(\frac{1}{2}\left(e+\frac{1}{e}\right)+\frac{1}{2}\left(e-\frac{1}{e}\right)\right) = \frac{e}{4}$$

Hence

$$f(m) = \frac{e}{4} - \frac{m}{2}(e-e^{-1}) + \frac{m^2}{2}(e+e^{-1}) = e\left(\frac{m^2}{2}-\frac{m}{2}+\frac{1}{4}\right) + e^{-1}\left(\frac{m^2+m}{2}\right)$$

and so $a_m = \frac{m^2+m}{2}$ and $b_m = 2m^2 - 2m + 1$. Clearly the limit of b_m/a_m is 4, and so our answer is $\boxed{8088}$.

15. In $\triangle ABC$, let G be the centroid and let the circumcenters of $\triangle BCG, \triangle CAG$, and $\triangle ABG$ be I, J, and K, respectively. The line passing through I and the midpoint of BC intersects KJ at Y. If the radius of circle K is 5, the radius of circle J is 8, and $AG = 6$, what is the length of KY?

Answer: $\mathbf{2 + \frac{\sqrt{55}}{2}}$

Solution: We see that both K and J are on the perpendicular bisector of AG, so KJ is the perpendicular bisector of AG. Let KJ intersect at AG at X. Since X is the midpoint of AG, we have $AX = 3$. Also, we can find that $KX = \sqrt{(AK)^2-(AX)^2} = \sqrt{25-9} = 4$ and $JX = \sqrt{(AJ)^2-(AX)^2} = \sqrt{64-9} = \sqrt{55}$. So, $KJ = KX + JX = 4 + \sqrt{55}$.

Now, we claim that Y is the midpoint of KJ. Let the midpoints of BC, CA, and AB be D, E, and F, respectively. Also, let O_B and O_C be the centers of the circumcircles of $\triangle AXF$ and $\triangle AXE$, respectively. We want to show that ID, the perpendicular bisector of BC passes through the midpoint of KJ.

This is equivalent to showing that the perpendicular bisector of FE passes through the midpoint of O_BO_C, since this configuration is the same as that we want to prove but scaled by a factor of $\frac{1}{2}$ toward point A.

Let Z be the midpoint of FE. Since AD is the median, Z lies on AD and thus is on line AX. So, Z is on the radical axis of $\odot O_B$ and $\odot O_C$. Let FE intersect $\odot O_B$ and $\odot O_C$. again at points Z_B and Z_C, respectively. Since Z lies on the radical axis, by Power of a Point we have $(ZF)(ZZ_B) = (ZE)(ZZ_C) \Rightarrow ZZ_B = ZZ_C \Rightarrow FZ_B = EZ_C$ since Z is the midpoint of FE.

If we project O_B and O_C onto Z_BZ_C, the images of O_B and O_C are the midpoints of FZ_B and EZ_C, respectively, so the midpoint of O_BO_C must be project to the midpoint of Z_BZ_C, which is also the midpoint of FE, thus proving our claim. This means that $KY = \frac{1}{2}KJ = \boxed{2 + \frac{\sqrt{55}}{2}}$.