Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise.
No calculators.

1. For all positive integers $n > 1$, let $f(n)$ denote the largest odd proper divisor of n (a proper divisor of n is a positive divisor of n except for n itself). Given that $N = 20^{23} \cdot 23^{20}$, compute
$$\frac{f(N)}{f(f(f(N)))}.$$

2. A 3×3 grid is to be painted with three colors (red, green, and blue) such that

   (i) no two squares that share an edge are the same color and

   (ii) no two corner squares on the same edge of the grid have the same color.

   As an example, the upper-left and bottom-left squares cannot both be red, as that would violate condition (ii). In how many ways can this be done? (Rotations and reflections are considered distinct colorings.)

3. How many trailing zeros does the value
$$300 \cdot 305 \cdot 310 \cdots 1090 \cdot 1095 \cdot 1100$$
end with?

4. Michelle is drawing segments in the plane. She begins from the origin facing up the y-axis and draws a segment of length 1. Now, she rotates her direction by $120°$, with equal probability clockwise or counterclockwise, and draws another segment of length 1 beginning from the end of the previous segment. She then continues this until she hits an already drawn segment. What is the expected number of segments she has drawn when this happens?

5. Ryan chooses five subsets S_1, S_2, S_3, S_4, S_5 of $\{1, 2, 3, 4, 5, 6, 7\}$ such that $|S_1| = 1, |S_2| = 2, |S_3| = 3, |S_4| = 4$, and $|S_5| = 5$. Moreover, for all $1 \le i < j \le 5$, either $S_i \cap S_j = S_i$ or $S_i \cap S_j = \emptyset$ (in other words, the intersection of S_i and S_j is either S_i or the empty set). In how many ways can Ryan select the sets?

6. We say that an integer $x \in \{1, \cdots, 102\}$ is *square-ish* if there exists some integer n such that $x \equiv n^2 + n \pmod{103}$. Compute the product of all square-ish integers modulo 103.

7. Let S be the number of bijective functions $f : \{0, 1, \ldots, 288\} \to \{0, 1, \ldots, 288\}$ such that $f((m+n) \bmod 17)$ is divisible by 17 if and only if $f(m) + f(n)$ is divisible by 17. Compute the largest positive integer n such that 2^n divides S.

8. Define the Fibonacci numbers via $F_0 = 0,\ F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$.

   Olivia flips two fair coins at the same time, repeatedly, until she has flipped a tails on both, not necessarily on the same throw. She records the number of pairs of flips c until this happens (not including the last pair, so if on the first flip both coins turned up tails c would be 0). What is the expected value of F_c?

9. Suppose a and b are positive integers with a curious property: $(a^3 - 3ab + \frac{1}{2})^n + (b^3 + \frac{1}{2})^n$ is an integer for at least 3, but at most finitely many different choices of positive integers n. What is the least possible value of $a + b$?

10. Colin has a peculiar 12-sided dice: it is made up of two regular hexagonal pyramids. Colin wants to paint each face one of three colors so that no two adjacent faces *on the same pyramid* have the same color. How many ways can he do this? Two paintings are considered identical if there is a way to rotate or flip the dice to go from one to the other. Faces are adjacent if they share an edge.