SMT 2023 Power Round

Preliminaries

Time Limit: 80 minutes.

Maximum Score: 166

Instructions: Problems that use the words "show", "prove", or "justify" require explanation or proof. Unless otherwise stated, full credit will be given for just the final answer. However, partial credit will be given for close attempts, so teams may find it in their best interest to show work regardless. Answers should be written on sheets of loose paper, clearly labeled, with every problem on its own sheet. Write the problem number in the top left corner. If you have multiple pages for a problem, number them and write the total number of pages for the problem in the bottom right corner (e.g. 1/2, 2/2).

Indicate your team ID number in the top right on each paper that you submit. Only submit one set of solutions for the team. Do not turn in any scratch work. In your solution for a given problem, you may cite the statements of earlier problems (but not later ones) without additional justification, even if you haven't solved them. The problems are ordered by content, NOT DIFFICULTY. It is to your advantage to attempt problems from throughout the test. While completing the round, you should not any materials outside of the content of this test (including results not covered in this power round). You may not use calculators. Good luck!

Grading Guide: points marked with (partial) should be given out if and only if **no** non-partial points were received. Multiple partial points can be given.

1 Introduction to Game Theory

This section focuses on providing an introduction to the fundamental concepts of game theory that we will build off in the later sections. Intuitively, games are situations where participants' payoffs (what they get out of a game, whether it be money, joy from winning, etc.) depend not only on their own actions but also on the actions of their opponents. Examples include games in the traditional sense such as chess and rock-paper-scissors but also can include real-world interactions such as auctions and how firms decide prices.

In any given game, there is a set of players (this test will focus on the case where there are only two players, but there can be any number of players in general); each player has a set of possible actions; and finally, there is some mapping that takes action profiles to payoffs.

We start by working through formalizing the game of rock-paper-scissors. Rock-paper-scissors is a game we are all familiar with, and we all probably have some intuitive idea about how to play it. How might we describe it mathematically?

- This is a game involving $n = 2$ players (we can have more than 2 players in general, but this section will focus on the case of two players).
- Each player is allowed to throw rock, paper, or scissors. We could write this as

  $$A_1 = A_2 = \{\text{rock, paper, scissors}\}$$

  where A_1 is the set of actions available to the first player and A_2 is the set of actions available to the second player. In the particular case of "Rock, Paper, Scissors", both players have the same options, namely rock, paper, or scissors. However, this is not generally true: in tic-tac-toe for instance, if one player claims a square then that square is no longer available to the other player.
- To fully describe what happens in a single instance of the rock-paper-scissors game, I could provide an element of $A = A_1 \times A_2$, which is the set of **action profiles**. For instance, (rock, paper) is an element of A and corresponds to player one throwing rock and player two throwing scissors.
- There's either a winner and a loser, or the players draw. For concreteness, we assign each outcome a value: 1 for a win, 0 for a draw, and -1 for a loss. (This is sometimes called the player's **utility** or **payoff**).

  In general, for each combination of actions, each player gets a certain utility value. We can represent this as a function $u_i : A \to \mathbb{R}$ for the i-th player.

  All in all, a **normal form game** refers to the triple $(N, \{A_i\}, \{u_i\})$ of some number of players, a set of actions for each player, and a utility function for each player. For two players, we can represent this information succinctly in a table (called the **payoff/utility matrix**): the rows correspond to player 1's actions, the columns correspond to player 2's actions, and the pairs describe player 1 and player 2's utilities, respectively. For instance, if player

	R	P	S
R	$(0,0)$	$(-1,1)$	$(1,-1)$
P	$(1,-1)$	$(0,0)$	$(-1,1)$
S	$(-1,1)$	$(1,-1)$	$(0,0)$

  one throws rock and player two throws paper, this corresponds to row R and column P. The entry there is $(-1,1)$ which corresponds to player one losing and receiving a payoff of -1 and player two winning and receiving a payoff of 1. This is exactly what happens in rock-paper-scissors, as paper does in fact beat rock. To see the connection between the payoff matrix and each player's utility function, note

  $$u_1((R,P)) = -1 \text{ and } u_2((R,P)) = 1.$$

With this in mind, we move on to what it means to "solve" a game. Intuitively, we want choices to be such that no single player can do better for themselves, keeping the choices of other players fixed.

Definition 1.1 (Pure Strategy Nash Equilibrium). *A pure strategy profile $\{a_i\}$ forms a Nash equilibrium of a game $(N, \{A_i\}, \{u_i\})$ if there are no profitable deviations: for all players j, their action a_j achieves a weakly higher payoff than any action a_j' keeping all other actions a_{-j} fixed. In other words, $u_j(a_j, a_{-j}) \geq u_j(a_j', a_{-j})$ for all a_j'.*

One method of computing Nash Equilibrium is by finding best responses. In particular, when given a payoff matrix, do the following:

1. For every column, circle player one's maximum attainable payoff in that column if player two plays the action of that column;
2. For every row, circle player two's maximum attainable payoff in that row if player one plays the action of that row;
3. If an entry in the payoff matrix has both player one's payoff and player two's payoff circled, then we call the action profile that leads to that payoff profile a *mutual best response*.

Going forward, call this process "circling best responses". Here is some practice for finding mutual best-responses.

Problem 1.1 (2 points)**.** *Find all strategy profiles that are mutual best responses of the following game:*

	X	Y
A	$(10; 7)$	$(5; 1)$
B	$(7; 8)$	$(2; 3)$

The best responses for player one are in red and the best responses for player two are in blue.

	X	Y
A	$(10; 7)$	$(5; 1)$
B	$(7; 8)$	$(2; 3)$

Then, the two mutual best responses are $(A; X)$ and $(B; Y)$.

- 1 points: $(A; X)$ is found
- 1 points: $(B; Y)$ is found
- 1 (partial) points: Best responses are found (best response to A is X and best response to B is Y; best response to X is A and best response to Y is B) or response lists $(10; 7)$ or $(2; 3)$

Problem 1.2 (3 points)**.** *Find all strategy profiles that are mutual best responses of the following game:*

	X	Y	Z
A	$(5; 3)$	$(1; 0)$	$(7; 1)$
B	$(5; 1)$	$(0; 0)$	$(7; 1)$
C	$(3; 10)$	$(1; 1)$	$(5; 7)$

The best responses for player one are in red and the best responses for player two are in blue.

	X	Y	Z
A	$(5; 3)$	$(1; 0)$	$(7; 1)$
B	$(5; 1)$	$(0; 0)$	$(7; 1)$
C	$(3; 10)$	$(1; 1)$	$(5; 7)$

Mutual best responses are $(B; X); (A; Y); (B; Z)$.

- 1 points: $(B; X)$ is found
- 1 points: $(A; Y)$ is found
- 1 points: $(B; Z)$ is found
- 1 (partial) points: Best responses are found or response lists payoffs instead of actions

Next, we show that the process of circling best responses finds all pure strategy Nash equilibria.

Problem 1.3 (4 points)**.** *Prove that the process of circling best responses finds exactly all Nash equilibria.*

Suppose an action profile is a mutual best response. Then, no player can get strictly higher from playing a different action as otherwise, their original action would not have been circled.
The converse. We will prove the contrapositive: if an action profile is not found by circling best responses, it is not a Nash equilibrium. If an action profile is not found by best responses, then there is at least one player who can do strictly better by deviating. As such, the action profile cannot be a Nash equilibrium.

- 2 points: One direction
- 2 points: Other direction

So far, the games in normal form we've seen only capture situations where moves are decided simultaneously by each player. Clearly, there are situations where this is not the case and timing plays a role. To capture this, we now introduce extensive form games. Extensive form games can be represented by game trees.

For example, consider the fight-or-flight game. Two animals encounter one another. The first animal chooses to either fight or flight. The second animal observes the first animal's choice and also decides to fight or flight. If both fight, both animals are injured. If both flight, nothing happens. If one fights and the other flights, the animal that chooses fight wins.

Let's try to first express this game as a normal form. There clearly are $n = 2$ players. The first animal's action space is $A_1 = f$fight, flight$g$. However, the second animal's action space is no longer just $f$fight, flight$g$: their decision is informed by observing what the first animal does. Intuitively, animal two makes two choices: what to do if animal one fights, and what to do if animal one flights. As such, there are four total strategies (two choices for the two actions animal one has):

1. fight if animal one fights and fight if animal one flights;
2. fight if animal one fights and flight if animal one flights;
3. flight if animal one fights and fight if animal one flights;
4. flight if animal one fights and flight if animal one flights.

To simplify notation, let the action $(x; y)$ denote "x if animal one fights and y if animal one flights". Thus, we have

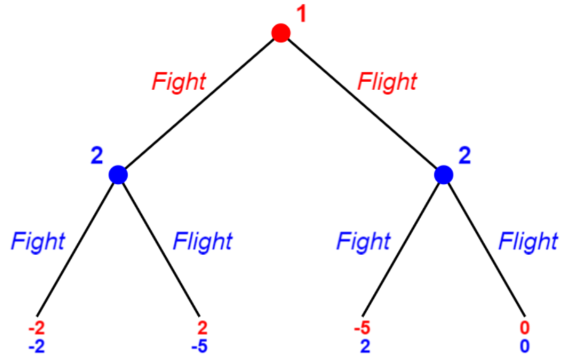
$$A_2 = f\text{(fight, fight); (fight, flight);(flight, fight); (flight, flight)}g:$$

Writing out the normal form will have $A_1 = f$fight, flight$g$ as rows and A_2 as defined above as columns. However, what are the payoffs?

If animal one plays flight and animal two plays (fight, flight) then animal one flights and animal two also flights, as their strategy of "fight if animal one fights, flight if animal one flights" leads to flight when animal one flights. We can model this with the following payoff matrix (with payoffs chosen arbitrarily):

	(fight,fight)	(fight,flight)	(flight,fight)	(flight,flight)
fight	$(\ 2; \ \ 2)$	$(\ 2; \ \ 2)$	$(2; \ \ 5)$	$(2; \ \ 5)$
flight	$(\ 5; 2)$	$(0; 0)$	$(\ 5; 2)$	$(0; 0)$

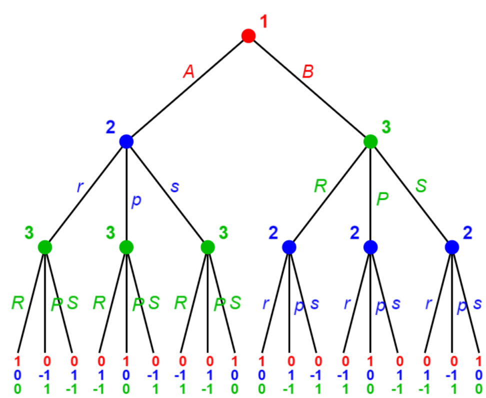
Alternatively, we can represent this game with a game tree:



To read this game tree, start from the top. At the node labeled 1, animal one has a choice of fight or flight, represented by the two edges. After player one's choice, animal two has a choice at each of the nodes labeled 2: one node for if player one chose fight and one node for if player two chose flight. At each of those nodes, animal two can choose to either fight or flight. Finally, payoffs are listed at the bottom corresponding to which path the animals' actions take.

Problem 1.4 (4 points)**.** *Draw a game tree for the following game:*

1. *Player three chooses between A and B;*
2. *If player one chooses A, then players one and two play rock-paper-scissors, except player two moves first and reveals what they play to player three before player two's choice;*
3. *If player one chooses B, then players two and three play rock-paper-scissors, except player three moves first and reveals what they play to player two before player two's choice;*
4. *Payo s are as follows:*
   - *Player one's payo is 1 if players two and three tie in their game of rock-paper-scissors and is 0 otherwise;*
   - *Player two's payo is 1 if they win the game of rock-paper-scissors, 1 if they lose the game of rock-paper-scissors, and 0 otherwise;*
   - *Player three's payo is 1 if they win the game of rock-paper-scissors, 1 if they lose the game of rock-paper-scissors, and 0 otherwise.*



- 1 points: Structure correct
- 1 points: Nodes labeled with correct player
- 1 points: Edges labeled with correct action
- 1 points: Payoffs correct

In general, extensive form games are:

Definition 1.2 (Extensive Form Game)**.** *An extensive form game represented by a game tree consists of the following:*

1. *A set of players $N = \{1, 2, \ldots, n\}$ as before;*
2. *A rooted tree[1] with edges E, non-terminal nodes D (also known as decision nodes, and hence the D) and terminal nodes T;*
3. *A function $P : D \to N$ that indicates which player moves at each non-terminal node;*
4. *A function $u : T \to \mathbb{R}^n$ that assigns a payoff to each terminal node.*

In this setting, the action space for player i at node v if $P(v) = i$ is the set of edges starting at v pointing away from the root of the game tree. Let $A(v)$ denote the possible actions for player $P(v)$ at node v.

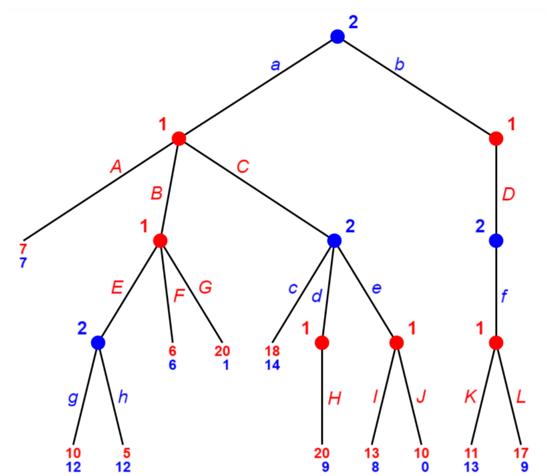
Since players now might need to make multiple decisions, the set of strategies they choose from is richer. Formally, we have the following.

Definition 1.3 (Strategy)**.** *A strategy for player i is a choice of action for every node they play at. Mathematically, player i's strategy space is the set of functions*

$$S_i = \{s_i : P^{-1}(i) \to A \text{ such that } s_i(v) \in A(v)\}.$$

Note: An action must be specified at every node, even nodes that are never reached due to some previous action. As such, player 2's strategy space in the extensive-form fight or flight game needs to specify what they do when player 1's strategy is fight **and** when player 1's strategy is flight, even though only one of the two will actually be played. As such, player 2's strategy space is {(flight, flight), (flight, fight);(fight, flight); (fight, fight)} opposed to {fight, flight}.

Problem 1.5 (2 points)**.** *Find the number of strategies in each player's strategy space.*



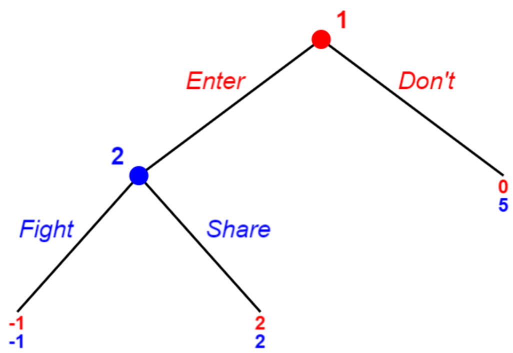
Player one chooses between $\{A, B, C\}$ and $\{D\}$; then between $\{E, F, G\}$; $\{H\}$; $\{I, J\}$; and $\{K, L\}$. Thus, they have a total of $3 \cdot 1 \cdot 3 \cdot 1 \cdot 2 \cdot 2 = 36$ possible strategies.
Player two chooses between $\{a, b\}$; then between $\{g, h\}$, $\{c, d, e\}$, and $\{f\}$. Thus, they have a total of $2 \cdot 2 \cdot 3 \cdot 1 = 12$ possible strategies.

- 1 points: Finds 36 strategies for player one
- 1 points: Finds 12 strategies for player two

In extensive games, the same definition of Nash Equilibrium holds. One issue when games are sequential is that some threats may not be credible. If one player's strategy specifies some choice at a future decision node, how do we know that they'll actually carry out that action when we arrive at that point?

Consider the following market entry game:

[1] A rooted tree is a set of nodes V and edges $E \subset V \times V$ such that:
(a) there is some node $r \in V$ that is designated as the root;
(b) for all $v \in V$, there exists a unique sequence $v_1, \ldots, v_n$ such that $v_1 = r, v_n = v$, and $(v_i, v_{i+1}) \in E$ for all $i = 1, \ldots, n-1$.



There are two firms. Firm two is already in the market for axes at UC Berkeley. Firm one is deciding whether or not to enter the market. If firm one enters, firm two can either fight by lowering prices, harming both firms. Otherwise, they can share the market. If firm one does not enter, firm two keeps the entire market.

Clearly, firm two would prefer for firm one to not enter the market. One possible strategy for firm two is to commit to fighting if firm one enters (and fight even if firm one does not). However, if firm one does choose to enter, firm two's best response is to Share.

Problem 1.6 (2 points). *Find all Nash Equilibrium of the Market Entry game.*

> The best response to "Enter" is "Share". Both "Fight" and "Share" are best responses to "Don't". The best response to "Fight" is "Don't" and the best response to "Share" is "Enter". Thus, the Nash Equilibrium are ("Don't","Fight"), ("Enter","Share"). Note that unlike the fight or flight game, Firm 2's strategies consist only of a single choice, as they do not make a move if Firm 1 chooses "Don't".
> - 1 points: Found ("Don't","Fight")
> - 1 points: Found ("Enter","Share")

To weed out non-credible threats, we need a finer notion of equilibrium.

Definition 1.4 (Subgame). *Let an extensive form game have players N, a rooted tree with edges E, non-terminal nodes D, and terminal nodes T, a player assignment function P, and a payo function u. Then, the subgame of that extensive form game starting at node d 2 D is the extensive form game with:*

1. *The same set of players N;*
2. *A rooted tree T^0 with root d, and all edges and nodes that are below d in the original tree;*
3. *Player Assignment function P^0 equal to P restricted to non-terminal nodes of T^0;*
4. *Payo function u^0 equal to u restricted to terminal nodes of T^0.*

Definition 1.5 (Subgame Perfect Nash Equilibrium). *A set of strategies for each player forms a Subgame Perfect Nash Equilibrium strategy profile if for every subgame, the strategy of each player restricted to the subgame forms a Nash Equilibrium.*

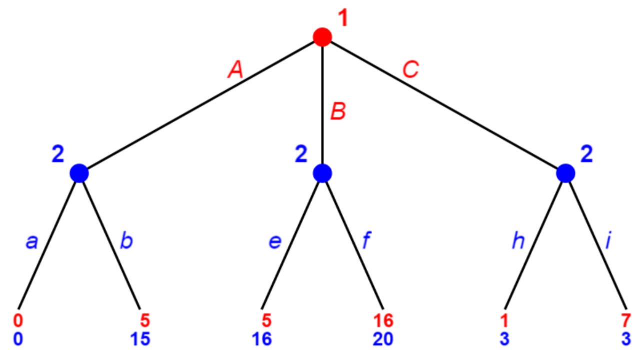
Problem 1.7 (2 points). *Find all Subgame Perfect Nash Equilibrium of the Market Entry game.*

> At the subgame where firm one plays "Enter", firm two must play "Share". Knowing that firm two will play "Share" if firm one plays "Enter", firm one will play "Enter" making ("Enter","Share") the subgame perfect equilibrium.
> - 2 points: Finding ("Enter","Share")
> - 1 (partial) points: Finding "Share" to be the equilibrium of the subgame after firm one plays "Enter"

Problem 1.8 (2 points). *Show that every Subgame Perfect Nash Equilibrium is a Nash Equilibrium. For full points, do not directly appeal to the definition of Nash Equilibrium.*

The entire game is a subgame of itself, so applying the definition of Subgame Perfect Nash Equilibrium to it gives that all subgame perfect Nash equilibrium are also Nash equilibrium.

- 2 points: Gives answer without using definition of Nash Eq
- 1 (partial) points: Gives answer using definition of Nash eq



Problem 1.9 (3 point). *Find the Subgame Perfect Nash Equilibrium of the above game.*

At the subgame after player one plays A, player two plays b. At the subgame after player one plays B, player two plays f. At the subgame after player one plays C, player two can either play h or i. If player one plays A they get a payoff of 5; if player one plays B they get a payoff of 16; and if player one plays i they get a payoff of either 1 or 7. Thus, player one maximizes their payoff by playing B. The two subgame perfect Nash equilibrium are $(B; bfh)$ and $(B; bfi)$.

- 1 points: $(B; bfh)$ is found
- 1 points: $(B; bfi)$ is found
- 1 points: Bonus point if both were found
- 1 (partial) points: All three subgames are analyzed
- 1 (partial) points: $(B; f)$ is found

Problem 1.10 (5 points). *Does every extensive form game with a finite number of moves have at least one Subgame Perfect Nash Equilibrium? Justify your answer.*

Yes. We will show this by induction on the number of moves in the longest possible path of the game.
If the longest path only has one move, then the game is essentially one player picking among a finite number of payoffs, which clearly has an equilibrium as the player just picks the largest payoff.
Suppose all games with longest paths of length n have subgame perfect Nash equilibrium, and consider any game with longest path of length $n + 1$. Consider all subgames that are achieved after n moves are played. There is only one person choosing a move at any of these subgames, so there is an equilibrium at each of these subgames (if there are multiple equilibrium, choose one arbitrarily). Replace the decision node at each of these subgames with a terminal node that has payoffs equal to the equilibrium payoff of the subgame. This is now a game with path length at most n and hence has a subgame perfect Nash equilibrium by the inductive hypothesis. This equilibrium along with the equilibrium actions of the subgames achieved after n moves forms a subgame perfect Nash equilibrium of the original game.

- 1 points: Correct answer of yes
- 2 points: Uses induction
- 2 points: Mathematically formal/rigorous

For the next problem, we need to specify the difference between a Subgame Perfect Nash Equilibrium and an outcome. A Subgame Perfect Nash Equilibrium specifies a full strategy profile for all players, including listing what happens at nodes that are never encountered. However, an outcome only lists the sequence of actions that are played. As such, multiple Subgame Perfect Nash Equilibrium (that differ on strategies at nodes that are never reached) can induce the same outcome.

Problem 1.11 (4 points)**.** *Does every extensive form game have at most one Subgame Perfect Nash Equilibrium outcome? Justify your answer.*

No. Consider the following game: Player one has two actions. Both actions give them the same payoff. Then, playing either action is a Subgame Perfect Nash Equilibrium and furthermore induce different outcomes.

- 1 points: Correct answer of no
- 3 points: Justification

To finish this section, we explore the connection between normal and extensive form games.

Definition 1.6 (Equivalent)**.** *A normal form game $(N; \{A_i\}; \{u_i\})$ is equivalent to an extensive form game with players N, player i's strategy space S_i, and player i's payoff function u_i' if for each $i \in N$ there exists a function h_i such that*

1. *h_i is a bijection from A_i to S_i;*
2. *for every action profile $a_1, \ldots, a_n$ of the normal form game it holds that*

$$u(a_1, \ldots, a_n) = u'(h_1(a_1), \ldots, h_n(a_n)).$$

Similarly, an extensive form game with players N, player i's strategy space S_i, and player i's payoff function u_i is equivalent to a normal form game $(N; A; u')$ if for each $i \in N$ there exists a function g_i such that

1. *g_i is a bijection from S_i to A_i;*
2. *for every strategy profile $s_1, \ldots, s_n$ of the extensive form game it holds that*

$$u(s_1, \ldots, s_n) = u'(g_1(s_1), \ldots, g_n(s_n)).$$

Finally, two normal form games or two extensive form games can similarly be equivalent if there exists a bijection between action or strategy spaces that preserve payoffs.

Problem 1.12 (4 points)**.** *Show that tic-tac-toe[2] is equivalent to the following:*

- *Two players take turns picking numbers from $\{1, 2, \ldots, 9\}$ without repetition;*
- *If one player has previously selected three numbers that add to exactly 15, the game ends and they win;*
- *If all numbers have been selected and no player has won, the game ends in a tie.*

Consider the following magic square:

2	7	6
9	5	1
4	3	8

Then, each possible winning combination of squares that can be chosen in tic-tac-toe is a winning combination of numbers that wins the game introduced in this problem. As such, biject each number with the square it is on according to the above diagram.

- 4 points: Correct
- 2 (partial) points: Decent attempt was made

Problem 1.13 (4 points)**.** *Show that a normal form game is equivalent to an extensive form game if and only if the extensive form game is equivalent to the normal form game.*

[2] Two players take turns choosing squares on a 3x3 grid without replacement. If one player has three squares in a row, column, or long diagonal, the game ends and they win. If all squares are chosen with no winner, the game ends in a tie.

We will use the fact that bijections are invertible.
Suppose the normal form extensive game $(N, \{A_i\}, \{u_i\})$ is equivalent to the extensive form game with with players N, player i's strategy space S_i, and player i's payoff function u'_i. Then for each i there exists a function h_i such that

1. h_i is a bijection from A_i to S_i;
2. for every action profile $a_1, \ldots, a_n$ of the normal form game it holds that

$$u(a_1, \ldots, a_n) = u'(h_1(a_1), \ldots, h_n(a_n)).$$

Define the function $g_i = h_i^{-1}$ for each i. As h_i is a bijection from A_i to S_i we have that g_i is a bijection from S_i to A_i. Furthermore,

$$u'_i(s_1, \ldots, s_n) = u'(h_1(h_1^{-1}(s_1), \ldots, h_n(h_n^{-1}(s_n)) = u(g_1(s_2), \ldots, g_n(s_n)).$$

Conversely, suppose the functions h_i makes an extensive form game equivalent to some normal form game. Then, the functions $g_i = h_i^{-1}$ makes the normal form game equivalent to the extensive form game.

- 1 points: Says inverse works for one direction
- 1 points: Says inverse works for other direction
- 2 points: Justifies that the inverse works (only once is needed)

Problem 1.14 (6 points). *Does every two-player extensive form game have an equivalent two player normal form game? Justify your answer.*

Yes. Suppose player one plays, then player two plays. For player one, their strategy space is just the set of actions available to them in the extensive form. Let this also be their set of available actions in the normal form, so $A_1 = S_1$. Define $h_1 : S_1 \to A_1$ by the identity: $h_1(a) = h_1(a)$.
For player two, their strategy space is a selection of some action for every possible action player two could have played. Let this be their action space for the normal form game, so $S_2 = A_2$. Once again, define $h_2 : S_2 \to A_2$ by the identity: $h_2(a) = h_2(a)$.
Finally, define u'_1 and u'_2 by $u'_1(a_1, a_2) = u_1(a_1, a_2)$ and $u'_2(a_1, a_2) = u_2(a_1, a_2)$. We have

$$u'_1(h_1(a_1), h_2(a_2)) = u_1(a_1, a_2), \quad u'_2(h_1(a_1), h_2(a_2)) = u_2(a_1, a_2).$$

- 1 points: Gives correct answer of yes.
- 2 points: Defines action spaces of the equivalent normal form game
- 2 points: Defines bijection that gives equivalence
- 1 points: Defines payoff function for the equivalent normal form game

Problem 1.15 (6 points). *Does every two-player normal form game have an equivalent two player extensive form game? Justify your answer.*

No. Consider the matching parity game: each player chooses a number 1 or 2. If both are odd or both are even, player one gets a payoff of one and player two gets a payoff of zero. If the two numbers have different parity, player two gets a payoff of one and player one gets a payoff of zero.
Then, each player's action space is $\{1, 2\}$. If player one goes first in the extensive game, they must have two choices. When player two moves, they have two choices for either of player one's moves. However, this means that player two's strategy space must have size four but there is no bijection between $\{1, 2\}$ and any set of size four.

- 1 points: Gives correct answer of no.
- 5 points: Justification.

2 Mixed Strategies

Some games do not have Nash equilibrium in pure strategies. In particular, the game of rock-paper-scissors does not have a pure strategy Nash equilibrium.

Problem 2.1 (2 points)**.** *Show that the game of rock-paper-scissors has no pure strategy Nash equilibrium.*

From before, we know that if some strategy profile is a Nash equilibrium, it will be found by circling best responses. However, circling best responses does not return anything, so there are no pure strategy Nash equilibria of rock paper scissors.

- 2 points: Solved it.

Intuitively, when players play rock-paper-scissors, they end up randomizing over what to play (try playing against your teammates). To formalize this, we define the following:

Definition 2.1 (Probability Distribution)**.** *A probability distribution over a set X is a function $p : X \to [0,1]$ such that*

$$\sum_{x \in X} p(x) = 1.$$

Intuitively, $p(x)$ is just the probability of event x happening. Let ΔX denote the set of all probability distributions over the set X.

Definition 2.2 (Mixed Strategy Extension of a Normal Form Game)**.** *Consider the normal form game $(N, \{A_i\}, \{u_i\})$. The mixed-strategy extension of this normal form game is the game with:*

- *Players N;*
- *Each player $i \in N$ has action space ΔA_i with p_i denoting a distribution in ΔA_i;*
- *Each player $i \in N$ has the payoff function*

$$U_j(p_1, \dots, p_I) = \mathrm{E}[u_j(a_1, \dots, a_I)]$$

where the expectation is with respect to $a_1 \sim p_1, \dots, a_I \sim p_I$.

In the two person case, if player one plays p_1 and player two plays p_2 then payoffs are

$$U_1(p_1, p_2) = \sum_{x \in A_1} \sum_{y \in A_2} p_1(x) p_2(y) u_1(x, y)$$

and

$$U_2(p_1, p_2) = \sum_{x \in A_1} \sum_{y \in A_2} p_1(x) p_2(y) u_2(x, y).$$

In general, if player one plays p_1 and the other players play p_{-1}, player one's payoffs are

$$U_1(p_1, p_{-1}) = \sum_{x \in A_1} p_1(x) U_1(\delta_x, p_{-1})$$

where δ_x is a point mass:

Definition 2.3 (Point Mass)**.** *Given a set X, a point mass distribution at $x \in X$ is denoted by $\delta_x(\cdot)$ and is defined by*

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$$

Definition 2.4 (Mixed Strategy Nash Equilibrium)**.** *A mixed strategy Nash Equilibrium of the game $(N, \{A_i\}, \{u_i\})$ is any pure strategy Nash Equilibrium of the Mixed Strategy Extension $(N, \{\Delta A_i\}, \{U_i\})$.*

Let's turn back to rock-paper-scissors for an example. One possible probability distribution over moves is to play rock, paper, and scissors each with probability $1/3$. It turns out that both players doing this is a Nash equilibrium.

Problem 2.2 (2 points)**.** *Suppose player one plays rock, paper, and scissors each with probability $1/3$. Show that player two can not get strictly higher expected payoff than also playing each with probability $1/3$.*

Suppose player two plays rock with probability r and paper with probability p. As such, they play scissors with probability $1 - p - r$. Player two's expected payoff is then

$$r\left(\frac{1}{3}\cdot 1 + \frac{1}{3}\cdot -1 + \frac{1}{3}\cdot 0\right) + p\left(\frac{1}{3}\cdot 1 + \frac{1}{3}\cdot -1 + \frac{1}{3}\cdot 0\right) + (1 - r - p)\left(\frac{1}{3}\cdot 1 + \frac{1}{3}\cdot -1 + \frac{1}{3}\cdot 0\right) = 0$$

so expected payoff for player two turns out to not depend on r, p at all. As such, no strict improvements can be made.

- 1 points: Writes out expected payoff for player two
- 1 points: Shows that no profitable deviation exists

What's the connection between mixed strategies and pure strategies? A valid probability distribution is to set $p(a) = 1$ for some $a \in X$ and $p(x) = 0$ for all $x \neq a$. Recall that these distributions are called point masses.

Naturally, if both players play point masses, payoffs in the mixed strategy extension are equal to payoffs in the original game. Formally, let $(N, \{\Delta A_i\}, \{U_i\})$ be the mixed-strategy extension of the game $(N, \{A_i\}, \{u_i\})$. Then, for any $a \in A_1, b \in A_2$, we have that

$$U_1(\delta_a, \delta_b) = u_1(a, b) \text{ and } U_2(\delta_a, \delta_b) = u_2(a, b).$$

Problem 2.3 (1 point)**.** *Prove the above result.*

By the definition of U_i, we get

$$U_1(\delta_a, \delta_b) = 1 \cdot 1 \cdot \delta_a(a)\delta_b(b)u_1(a, b) + 0 = u_1(a, b).$$

Case for player two follows similarly.

- 1 points: Got it.

Problem 2.4 (6 points)**.** *Suppose (p_1, p_{-1}) is a mixed strategy Nash equilibrium of the game $(N, \{A_i\}, \{u_i\})$ and $p_1(a) > 0, p_1(b) > 0$. Show the following holds:*

$$U_1(\delta_a, p_{-1}) = U_1(\delta_b, p_{-1}) = U_1(p_1, p_{-1}).$$

Towards a contradiction, suppose $U_1(\sigma_a, p_{-1}) \neq U_1(p_1, p_{-1})$. If $U_1(\sigma_a, p_{-1}) > U_1(p_1, p_{-1})$ then player one has a profitable deviation from p_1 to σ_a so the original strategies are not a Nash equilibrium. If $U_1(\sigma_a, p_{-1}) < U_1(p_1, p_{-1})$ then player one has a profitable deviation from p_1 to p_1' defined as follows (other deviations may also work):

$$p_1'(x) = \begin{cases} \frac{p_1(x)}{1-p_1(a)} & \text{if } x \neq a \\ 0 & \text{if } x = a. \end{cases}$$

First, we check this is a distribution. As $p_1(x) \geq 0$ and $1 - p_1(a) \geq p_1(b) > 0$, we have that $p_1'(x) \geq 0$ for all x. Next,

$$\sum_{x \in A_1} \frac{p_1(x)}{1-p_1(a)} = \frac{1-p_1(a)}{1-p_1(a)} = 1.$$

Now, we show that player one has a profitable deviation from p_1 to p_1'. We have

$$\begin{aligned} U_1(p_1', p_{-1}) &= \sum_{x \in A_1} p_1'(x) U_1(\sigma_x, p_{-1}) \\ &= \sum_{x \neq a} \frac{p_1(x)}{1-p_1(a)} U_1(\sigma_x, p_{-1}) \\ &= \sum_{x \neq a} \left(p_1(x) + \frac{p_1(x)p_1(a)}{1-p_1(a)} \right) U_1(\sigma_x, p_{-1}) \\ &= \sum_{x \neq a} p_1(x) U_1(\sigma_x, p_{-1}) + \sum_{x \neq a} \frac{p_1(x)p_1(a)}{1-p_1(a)} U_1(\sigma_x, p_{-1}) \\ &= U_1(p_1, p_{-1}) - p_1(a)U_1(\sigma_a, p_{-1}) + \frac{p_1(a)}{1-p_1(a)}(U_1(p_1, p_{-1}) - p_1(a)U_1(\sigma_a, p_{-1})) \\ &= U_1(p_1, p_{-1}) + \frac{p_1(a)}{1-p_1(a)}(U_1(p_1, p_{-1}) - U_1(\sigma_a, p_{-1})) \\ &> U_1(p_1, p_{-1}). \end{aligned}$$

Thus, $U_1(\sigma_a, p_{-1}) = U_1(p_1, p_{-1})$. Replacing a with b above also shows that $U_1(\sigma_b, p_{-1}) = U_1(p_1, p_{-1})$.

- 1 points: Claimed there would be a profitable deviation otherwise
- 2 points: Constructed valid profitable deviation
- 3 points: Showed that the profitable deviation is in fact a profitable deviation

Problem 2.5 (3 points)**.** *Find the mixed strategy Nash Equilibrium of the normal form game with payoff matrix:*

	X	Y
A	$(10, 8)$	$(4, 1)$
B	$(2, 3)$	$(5, 7)$

By the previous problem, for (p_1, p_2) to be a Nash equilibrium we need $U_1(\sigma_A, p_2) = U_1(\sigma_B, p_2)$ which gives that
$$10p_2(X) + 4p_2(Y) = 2p_2(X) + 5p_2(Y).$$
We can then write $p_2(Y) = 1 - p_2(X)$ to get
$$\begin{aligned} & 10p_2(X) + 4(1 - p_2(X)) = 2p_2(X) + 5(1 - p_2(X)) \\ \implies & 4 + 6p_2(X) = 5 - 3p_2(X) \\ \implies & 9p_2(X) = 1 \\ \implies & p_2(X) = 1/9. \end{aligned}$$
Similarly,
$$\begin{aligned} & U_2(p_1, \sigma_X) = U_2(p_1, \sigma_Y) \\ \implies & 8p_1(A) + 3p_1(B) = 1p_1(A) + 7p_1(B) \\ \implies & 8p_1(A) + 3(1 - p_1(A)) = 1p_1(A) + 7(1 - p_1(A)) \\ \implies & 3 + 5p_1(A) = 7 - 6p_1(A) \\ \implies & 11p_1(A) = 4 \\ \implies & p_1(A) = 4/11. \end{aligned}$$
Thus, the mixed strategy Nash equilibrium is:$p_1(A) = 4/11, p_1(B) = 7/11, p_2(X) = 1/9, p_2(Y) = 8/9.$

- 3 points: Correct mixed strategy Nash equilibrium is found (finding a different mixed strategy Nash equilibrium in point masses is fine, but the equilibrium must be expressed in point masses: (σ_A, σ_X) instead of (A, X)
- 1 (partial) points: Tried to apply indifference condition from previous problem

Problem 2.6 (4 points). *Find the mixed strategy Nash Equilibrium of the normal form game with payoff matrix:*

	X	Y	Z
A	$(2, -3)$	$(-1, 0)$	$(7, -1)$
B	$(1, -1)$	$(0, 5)$	$(7, 1)$
C	$(3, 10)$	$(1, 4)$	$(5, 7)$

Since every pair of actions must generate the same expected payoff, all three actions must also generate the same expected payoff. Thus, for (p_1, p_2) to be a mixed strategy Nash equilibrium, we need
$$U_1(\sigma_A, p_2) = U_1(\sigma_B, p_2) = U_1(\sigma_C, p_2)$$
and
$$U_2(p_1, \sigma_X) = U_2(p_1, \sigma_Y) = U_2(p_1, \sigma_Z).$$
This translates into
$$2p_2(X) - 1p_2(Y) + 7p_2(Z) = 1p_2(X) + 0p_2(Y) + 7p_2(Z) = 3p_2(X) + 1p_2(Y) + 5p_2(Z)$$
and
$$3p_1(A) - 1p_1(B) + 10p_1(C) = 0p_1(A) + 5p_1(B) + 4p_1(C) = -1p_1(A) + 1p_1(B) + 7p_1(C).$$
Doing the algebra gives equilibrium strategies to be $p_1(A) = 2/5. p_1(B) = 1/5, p_1(C) = 2/5, p_2(X) = 2/7, p_2(Y) = 2/7, p_2(Z) = 3/7.$

- 4 points: Correct mixed strategy Nash equilibrium is found
- 2 (partial) points: Recognized indifference between all three actions

Next, we investigate the question of what strategies might be played in equilibrium. On the opposite end of the spectrum from best responses are dominated actions.

Definition 2.5 (Dominated Action). *An action a_i is a dominated action for player i of the mixed strategy extension $(N, \{\Delta A_i\}, \{U_i\})$ if for all possible mixed strategies of the other players p_{-i}, there exists p_i such that*

$$U_i(p_i, p_{-i}) > U_i(\delta_{a_i}, p_{-i}).$$

If this is the case, we also say that action a_i is dominated by the strategy p_i.

Problem 2.7 (4 points). *Suppose action a_i is dominated by some strategy p_i such that $p_i(a_i) > 0$ (action a_i is played with positive probability in strategy p_i). Show that action a_i is dominated by some strategy p_i' such that $p_i'(a_i) = 0$ (action a_i is never played in strategy p_i').*

> Define the strategy p_i' by
>
> $$p_i'(x) = \begin{cases} \frac{p_1(x)}{1-p_1(a)} & \text{if } x \neq a \\ 0 & \text{if } x = a. \end{cases}$$
>
> By the reasoning done in Problem 2.1, this strategy achieves higher expected payoff than strategy p_i for any opponent strategies, and hence also dominates action a_i.
> - 2 points: Finds a strategy p_i' that works
> - 2 points: Justifies that p_i' works

Problem 2.8 (6 points). *Show that an action a_i is a dominated action if and only if it is never played with positive probability in any best response to any opponent strategy.*

> Suppose a_i is a dominated action. Then, there exists some strategy p_i such that p_i is better than a_i for any opponent strategy and $p_i(a_i) = 0$. For any mixed strategy that plays a_i with positive probability, playing p_i instead of a_i will achieve a strictly higher payoff so no mixed strategy that plays a_i with positive probability is a best response.
> Suppose a_i is not a dominated action. Then, there exists some opponent strategy p_{-i} such that
>
> $$U_i(\delta_{a_i}, p_{-i}) \geq U_i(p_i, p_{-i})$$
>
> for all player i strategies p_i. Thus, always playing a_i is a best response to p_{-i}.
> - 3 points: One direction
> - 3 points: Other direction

One of the foundational results in game theory is that in any finite normal form game, there exists a Nash equilibrium in mixed strategies. While this result in full generality is beyond the scope of this round, we will prove a limited result in the case where there are two players, each player has two actions, and the game is zero-sum.

Definition 2.6 (Zero-Sum Game). *A two-player game is zero-sum if for all action profiles a, we have that $u_1(a) + u_2(a) = 0$.*

Suppose player one can play a_1 or a_2 and player two can play b_1 or b_2. Then, the payoff matrix for this game can be written as

	b_1	b_2
a_1	$(v_{11}, -v_{11})$	$(v_{12}, -v_{12})$
a_2	$(v_{21}, -v_{21})$	$(v_{22}, -v_{22})$

For the remainder of this section, this will be the game under consideration. In this case, each mixed strategy can be represented by a single value: q_1 for player one and q_2 for player two so $p_1(a_1) = q_1, p_1(a_2) = 1 - q_1$ and $p_2(b_1) = q_2, p_2(b_2) = 1 - q_2$. Going forward, let "the mixed strategy q_1" denote the mixed strategy $p_1(a_1) = q_1, p_1(a_2) = 1 - q_1$ and "the mixed strategy q_2" denote the mixed strategy $p_2(b_1) = q_2, p_2(b_2) = 1 - q_2$.

We will use the following facts to establish existence of Nash equilibrium in this case. Formal definitions for "continuous", "convex", and "concave" will be introduced when needed for problems.

Fact 2.1 (Continuous Functions attain Max/Min). *Suppose X and Y are closed intervals of $\mathbb{R}$ and f is a continuous function that maps $X \times Y$ to $\mathbb{R}$. Then, there exists $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that (x_1, y_1) solves*

$\max_x \min_y f(x, y)$ *and* (x_2, y_2) *solves* $\min_y \max_x f(x, y)$.

Fact 2.2 (Min-Max Theorem)**.** *Suppose* X *and* Y *are closed intervals and* f *is a continuous function* $X \times Y \to \mathbb{R}$ *such that*

- *holding* $\hat{y}$ *fixed,* $f(x, \hat{y})$ *is convex in* x *for all* $\hat{y}$*;*
- *holding* $\hat{x}$ *fixed,* $f(\hat{x}, y)$ *is concave in* y *for all* $\hat{x}$*.*

Then,

$$\max_x \min_y f(x, y) = \min_y \max_x f(x, y).$$

Here, $\max_x \min_y f(x, y)$ should be interpreted as "x is chosen first to maximize the smallest value $f(x, y)$ can take over all possible y values" while $\min_y \max_x f(x, y)$ should be interpreted as "y is chosen first to minimize the largest value $f(x, y)$ can take over all possible x values".

Overall, the proof will proceed as follows: first, we will show that Min-Max Theorem can be applied to this setting. Then, we will use the Min-Max Theorem to find the values that achieve the min max. Finally, we show that those values form a Nash Equilibrium.

Problem 2.9 (1 point)**.** *Identify each player's set of possible mixed strategies with a closed interval.*

Each player's mixed strategy space can be pinned down by the probability they play their first action, which is just a number in $[0, 1]$. This is clearly a closed and bounded interval.

- 1 points: They got it

Problem 2.10 (3 points)**.** *Suppose player one plays the mixed strategy* q_1 *and player two plays the mixed strategy* q_2*. What is the expected payoff for player one in terms of* $v_{11}, v_{12}, v_{21}, v_{22}$*?*

$$q_1 q_2 v_{11} + q_1(1 - q_2)v_{12} + (1 - q_1)q_2 v_{21} + (1 - q_1)(1 - q_2)v_{22}$$

- 3 points: They got it

Going forward, let $V(q_1, q_2)$ denote the expected payoff for player one when player one plays the mixed strategy q_1 and player two plays the mixed strategy q_2.

Definition 2.7 (Continuous)**.** *A function* $f : X \times Y \to \mathbb{R}$ *is continuous if for every* $(x, y) \in X \times Y$ *and* $\epsilon > 0$*, there exists* $\delta > 0$ *such that if the Euclidean distance between* (x', y') *and* (x, y) *is less than* δ*, then* $|f(x', y') - f(x, y)| < \epsilon$*.*

Problem 2.11 (6 points)**.** *Show that* $V(q_1, q_2)$ *is continuous.*

Fix (q_1, q_2) and let $\epsilon > 0$. Take $\delta = \frac{\epsilon}{v_{11}+v_{12}+v_{21}+v_{22}}$ and suppose the distance between (q_1', q_2') and (q_1, q_2) is less than δ. In particular, this implies that $|q_1' - q_1| < \delta$ and $|q_2' - q_2| < \delta$. Then,

$$\begin{aligned}
|f(q_1', q_2') - f(q_1, q_2)| \le\ & |f(q_1', q_2') - f(q_1, q_2')| + |f(q_1, q_2') - f(q_1, q_2)| \\
=& |(q_1' - q_1)(q_2' v_{11} + (1 - q_2')v_{12}) + (q_1 - q_1')(q_2' v_{21} + (1 - q_2')v_{22})| \\
&+ |(q_2' - q_2)(q_1 v_{11} + (1 - q_1)v_{21}) + (q_2 - q_2')(q_1 v_{12} + (1 - q_1)v_{22}| \\
\le\ & \delta(v_{11} + v_{12} + v_{21} + v_{22}) \\
<\ & \epsilon
\end{aligned}$$

as desired.

- 2 points: Chose an δ that worked
- 4 points: Justified it worked

Definition 2.8 (Convex/Concave)**.** *A function* $f : X \to \mathbb{R}$ *is convex if for all* $t \in [0, 1]$ *and* $x, x' \in X$ *it holds that*

$$f(tx + (1 - t)x') \le tf(x) + (1 - t)f(x').$$

A function $g : Y \to \mathbb{R}$ *is concave if* $-g$ *is convex.*

Problem 2.12 (6 points). *Show that $V(q_1, q_2)$ is convex in q_1 holding q_2 fixed and is also concave in q_2 holding q_1 fixed.*

Note that we can re-write player one's expected utility as

$$q_1(q_2v_{11}(1-q_2)v_{12}-q_2v_{21}-(1-q_2)v_{22})+q_2v_{21}+(1-q_2)v_{22}.$$

Let $a = q_2v_{11}(1-q_2)v_{12}-q_2v_{21}-(1-q_2)v_{22}$ and $b = q_2v_{21}+(1-q_2)v_{22}$ so expected utility is just $q_1 \cdot a + b$. Let $f(q_1) = q_1 \cdot a + b$; we'll show this function is convex:

$$f(tq_1+(1-t)q_1') = (t \cdot q_1 + (1-t) \cdot q_1')a + b = t(q_1 \cdot a + b) + (1-t)(q_1' \cdot a + b) = tf(q_1) + (1-t)f(q_1').$$

For concavity, we can either repeat the same argument, re-writing $V(q_1, q_2) = q_2 \cdot m + n$ for some m, n or by noting that the original expected profit function is symmetric.

- 6 points: Multiple ways to show this. Maybe 3 points if the idea is there and 3 points if the execution is there.

Problem 2.13 (4 points). *Show that there exists q_1, q_2 such that*

$$V(q_1, q_2) = \max_{q_1}\min_{q_2} V(q_1, q_2) = \min_{q_2}\max_{q_1} V(q_1, q_2).$$

By the previous three problems, we can apply the Min-Max Theorem to $V : [0,1] \times [0,1] \to \mathbb{R}$. Then,

$$\max_{q_1}\min_{q_2} V(q_1, q_2) = \min_{q_2}\max_{q_1} V(q_1, q_2).$$

By Fact 2.1, there exists q_1, q_2 such that

$$f(q_1, q_2) = \max_{q_1}\min_{q_2} V(q_1, q_2).$$

By the Min-Max Theorem,

$$f(q_1, q_2) = \max_{q_2}\min_{q_1} V(q_1, q_2)$$

as well.

- 1 points: States conditions of Min-Max Theorem are satisfies
- 1 points: Applies Fact 2.1
- 2 points: Applies the Min-Max Theorem

Problem 2.14 (10 points). *Show that q_1, q_2 from the previous problem is a Nash Equilibrium of the two-player two-action zero-sum game.*

We will show that there are no profitable deviations for either player. First, suppose player one has a profitable deviation from q_1 to q_1'. Then,

$$V(q_1', q_2) > V(q_1, q_2) = \min_{q_2}\max_{q_1} V(q_1, q_2).$$

However, this contradicts the fact that q_2 was chosen to minimize the maximal value of $V(q_1, q_2)$ since a different choice of q_1 now allows V to be greater than $\min_{q_2}\max_{q_1} V(q_1, q_2).$

Next, suppose player two has a profitable deviation from q_2 to q_2'. As player two's utility function is $-V$, this means that

$$V(q_1, q_2') < V(q_1, q_2) = \max_{q_1}\min_{q_2} V(q_1, q_2).$$

However, this contradicts the fact that q_1 was chosen to maximize the minimal value of $V(q_1, q_2)$ since a different choice of q_2 now allows V to be lesser than $\max_{q_1}\min_{q_2} V(q_1, q_2).$

- 5 points: no profitable deviations for player one
- 5 points: no profitable deviations for player two

3 Application to Auctions and Competition

3.1 Second Price Auctions

The first application of game theory we analyze are second price auctions. The setting is as follows:

- There is some set of agents $1, 2, \ldots, n$ and a single good;
- Each agent i values the good at v_i for $i = 1, 2, \ldots, n$ (you may assume that all v_i's are distinct);
- Each agent i bids some value b_i for $i = 1, 2, \ldots, n$.

Each agent only knows their own value and bids are made simultaneously. After all bids are made, the agent that submitted the highest bid wins the item and pays the second highest bid. All other agents do not receive the item and pay nothing. If agent i obtains the object at price p, their utility is $v_i - p$. Otherwise, if agent i does not the obtain the object and does not pay anything, their utility is 0.

Problem 3.1 (2 points)**.** *Write agent i's utility as a function of $v_i, b_1, b_2, \ldots, b_n$. In other words, define $u_i(b_1, \ldots, b_n)$.*

$$u_i(b_1, \ldots, b_n) = \begin{cases} v_i - \max\{b_j : j \neq i\} & \text{if } b_i = \operatorname{argmax}\{b_1, \ldots, b_n\} \\ 0 & \text{otherwise} \end{cases}.$$

- 2 points: they get it

Problem 3.2 (3 points)**.** *Suppose agent i wins the item and pays $p > v_i$. Show that agent i has a profitable deviation or prove that they do not.*

Agent i's payoff is $v_i - p < 0$. They can deviate to bidding v_i: if they still win then the new price they pay must be less than or equal to v_i so their payoff is non-negative. If they lose, their payoff increases to zero.

- 1 points: Describes a profitable deviation.
- 2 points: Shows the deviation is in fact profitable.

Problem 3.3 (4 points)**.** *Suppose agent i wins the item and pays $p \leq v_i$. Show that agent i has a profitable deviation or prove that they do not.*

Currently, their payoff is $v_i - p \geq 0$. If they deviate and end up losing the auction, their payoff is 0 which cannot be profitable. If they deviate and still win, then the second highest bid is still unchanged and thus the price they pay is also unchanged. Thus, their utility remains $v_i - p$. In either case, they do not have a profitable deviation.

- 2 points: Shows that deviating to losing cannot be profitable
- 2 points: Shows that any deviation that remains winning cannot be profitable

Problem 3.4 (5 points)**.** *Describe a Nash equilibrium of the second price auction. Justify that it is a Nash equilibrium.*

Each player bids their own valuation. If they win, they pay less than what they bid, so they have no incentive to deviate. If they lose, any deviation to where they win must make them pay more than their valuation, so there are no profitable deviations.

- 2 points: Gives the correct Nash equilibrium
- 3 points: Justification

3.2 First Price Auctions

First price auctions share the same environment as second price auctions. The one difference is that instead of paying the second highest bid, the winner of the action (still the agent with the highest bid) pays their own bid. Payoffs are still the same. While calculus is needed to derive the Nash equilibrium of first price auctions, we can still investigate them a bit.

Problem 3.5 (2 points)**.** *Write agent i's utility as a function of $v_i, b_1, b_2, \ldots, b_n$. In other words, define $u_i(b_1, \ldots, b_n)$.*

$$u_i(b_1, \ldots, b_n) = \begin{cases} v_i - b_i & \text{if } b_i = \operatorname{argmax}\{b_1, \ldots, b_n\} \\ 0 & \text{otherwise}. \end{cases}$$

- 2 points: they get it

Problem 3.6 (3 points)**.** *Suppose agent i wins the item and pays $p = v_i$ and there is some distance between each bid. Show that agent i has a profitable deviation or prove that they do not.*

Since there is some distance between each bid, agent i can decrease their bid by a sufficiently small amount to still win but pay a bit less.

- 2 points: Gets deviation is to decrease their bid a bit
- 1 points: Argues that the deviation is profitable

3.3 Competition

We now turn our attention to competition between firms. The setting here is as follows:

- There are two firms, each firm $i = 1, 2$ has a cost function $c_i(q_i)$ that describes the cost to firm i to produce q_i goods;
- There is an inverse demand function $p(q)$ that describes the market price of each good when there are q total goods in the market;
- Each firm's utility function is their profit:

$$u_i(q_i, q_{-i}) = p(q_i + q_{-i}) \cdot q_i - c_i(q_i)$$

for $i = 1, 2$.

Suppose $p(q) = 22 - 2q$, $c_1(q_1) = 6q_1$, and $c_2(q_2) = 2q_2.$ The following fact may be useful:

Fact 3.1 (Maximum of a Quadratic)**.** *The function $f(x) = -(x - a)(x - b)$ attains a maximum at $x = \frac{a+b}{2}$.*

Problem 3.7 (3 points)**.** *What is firm one's best response to firm two playing $\bar{q}_2$?*

Keeping $q_2 = \bar{q}_2$, firm one's utility is

$$u_1(q_1, q_2 = \bar{q}_2) = (22 - 2(q_1 + \bar{q}_2))q_1 - 6q_1 = q_1(16 - 2q_1 - 2\bar{q}_2) = -2q_1(q_1 - (8 - \bar{q}_2))$$

so by the hint, attains a maximum at $q_1 = 4 - \bar{q}_2/2$.

- 3 points: Gets correct best response
- 1 (partial) points: Gets setup

Problem 3.8 (3 points)**.** *What is firm two's best response to firm one playing $\bar{q}_1$?*

Fixing $q_1 = \bar{q}_1$, firm two's utility is

$$u_2(q_1 = \bar{q}_1, q_1) = (22 - 2(\bar{q}_1 + q_2))q_2 - 2q_2 = q_2(20 - 2\bar{q}_1 - 2q_2) = -2q_2(q_2 - (10 - \bar{q}_1))$$

so by the hint, attains a maximum at $q_2 = 5 - \bar{q}_1/2$.

- 3 points: Gets correct best response
- 1 (partial) points: Gets setup

Problem 3.9 (4 points)**.** *Find the Nash Equilibrium of the competition game if firms choose quantities simultaneously. What are equilibrium profits?*

In equilibrium, both quantities must be best responses to one another. Thus, $q_1 = 4 - q_2/2$ and $q_2 = 5 - q_1/2$. Plugging one into the other gives

$$q_1 = 4 - (5 - q_1/2)/2 \implies q_1 = 2.$$

Then,

$$q_2 = 5 - q_1/2 = 5 - 2/2 = 4.$$

Thus, the Nash equilibrium is for firm one to produce $q_1 = 2$ and firm two to produce $q_2 = 4$.
In equilibrium, the price is $22 - 2(2 + 4) = 10$. Firm one makes a profit of $10 \cdot 2 - 6 \cdot 2 = 8$ and firm two makes a profit of $10 \cdot 4 - 2 \cdot 4 = 32$.

- 3 points: Gets correct Nash equilibrium
- 1 points: Gets correct profits
- 2 (partial) points: Correct setup

Problem 3.10 (4 points). *Find the Subgame Perfect Nash Equilibrium of the competition game if firm one chooses their quantity before firm two (and firm two can observe their choice before choosing their own quantity).*

Note that as firm two observes firm one's quantity, their strategy will be a response for every possible quantity firm one chooses. By Problem 3.8, firm two will always play $q_2(q_1) = 5 - q_1/2$. Firm one thus seeks to maximize

$$u_1(q_1 | q_2 = 5 - q_1/2) = (22 - 2(q_1 + 5 - q_1/2))q_1 - 6q_1 = (12 - q_1)q_1 - 6q_1 = -q_1(q_1 - 6)$$

which attains a maximum at $q_1 = 3$ by the hint. Thus, the Nash equilibrium is firm one produces $q_1 = 3$ and firm two produces $q_2 = 5 - q_1/2.$
In equilibrium, firm two will end up producing $q_2 = 5 - 3/2 = 3.5.$ Equilibrium price is $22 - 2(3 + 3.5) = 9$. Profit for firm one is $9 \cdot 3 - 6 \cdot 3 = 9$ and profit for firm two is $9 \cdot 3.5 - 2 \cdot 3.5 = 24.5$.

- 3 points: Gets correct Nash equilibrium (note: $(q_1 = 3, q_2 = 3.5)$ is what *happens* in Nash equilibrium, but is not the Nash equilibrium itself)
- 1 points: Gets correct profits
- 1 (partial) points: Correct setup
- 1 (partial) points: Puts $(q_1 = 3, q_2 = 3.5)$ as the Nash equilibrium

Problem 3.11 (4 points). *Find the Subgame Perfect Nash Equilibrium of the competition game if firm two chooses their quantity before firm one (and firm one can observe their choice before choosing their own quantity).*

By Problem 3.7, firm one will always play $q_1(q_2) = 4 - q_2/2$. Firm two thus seeks to maximize

$$u_2(q_2 | q_1 = 4 - q_2/2) = (22 - 2(q_2 + 4 - q_2/2))q_2 - 2q_2 = (14 - q_2)q_2 - 2q_2 = -q_2(q_2 - 12)$$

which attains a maximum at $q_2 = 6$ by the hint. Thus, the Nash equilibrium is firm two produces $q_2 = 6$ and firm one produces $q_1 = 4 - q_2/2.$
In equilibrium, firm one will end up producing $q_1 = 4 - 6/2 = 1.$ Equilibrium price is $22 - 2(6 + 1) = 8$. Profit for firm one is $8 \cdot 1 - 6 \cdot 1 = 2$ and profit for firm two is $8 \cdot 6 - 2 \cdot 6 = 36$.

- 3 points: Gets correct Nash equilibrium (note: $(q_1 = 1, q_2 = 6)$ is NOT the Nash equilibrium)
- 1 points: Gets correct profits
- 1 (partial) points: Correct setup
- 1 (partial) points: Gets $(q_1 = 1, q_2 = 6)$ as the Nash equilibrium

3.4 We Scream for Ice Cream

Congratulations on making it to the end of the power round! To celebrate, all 500 SMT participants go to the beach, which happens to be 500 meters long so there is one person every meter. At each end of the beach is an ice cream stand, suppose stand A is at meter 0 and stand B is at meter 500 on the beach. Stand A charges price p_A and stand B charges price p_B. The overall cost to buy ice cream from a stand is equal to the price charged by the stand plus the distance to the stand (so for someone at meter 50, the cost to buy ice cream from stand A is $50 + p_A$ and the cost to buy ice cream from stand B is $450 + p_B$). Each of the 500 SMT participants buys ice cream from the stand that has a lower overall cost. Each ice cream stand's payoff is equal to the price they charge multiplied by the number of people they serve.

Problem 3.12 (2 points)**.** *At what meter is the cost to going to stand A equal to the cost of going to stand B if stand A charges p_A and stand B charges p_B?*

Let the indifferent meter be d. At this point, the cost of going to either stand is the same, so $p_A + m = p_B + 500 \quad d$: Solving for d gives $d = \frac{p_B \quad p_A + 500}{2}$.

- 2 points: Gets correct answer
- 1 (partial) points: Sets up the indifference condition

Problem 3.13 (2 points)**.** *Write each stand's profits as a function of p_A and p_B.*

For stand A, everyone at meter less than d goes to their stand, so their profit is

$$u_A(p_A; p_B) = p_A \quad d = p_A \quad \frac{p_B \quad p_A + 500}{2}:$$

For stand B, everyone at meter greater than d goes to their stand, so their profit is

$$u_B(p_A; p_B) = p_B \quad (500 \quad d) = p_B \quad \frac{p_A \quad p_B + 500}{2}:$$

- 1 points: Found profit for stand A
- 1 points: Found profit for stand B

Problem 3.14 (6 points)**.** *What prices do the two firms charge in equilibrium?*

First, we find best responses. Re-writing profit functions gives

$$u_A(p_A; p_B) = \quad \frac{1}{2} p_A (p_A \quad (p_B + 500))$$

and

$$u_B(p_A; p_B) = \quad \frac{1}{2} p_B (p_B \quad (p_A + 500)):$$

Thus, stand A's best response to p_B is $250 + p_B = 2$ while stand B's best response to p_A is $250 + p_A = 2$. For the two stands to be best responding to one another, we have $p_A = p_B = 500$.

- 6 points: Found Nash equilibrium
- 4 (partial) points: Found best responses

Problem 3.15 (8 points)**.** *Suppose a better wooden boardwalk on the beach decreases the cost of walking, making the overall cost of buying ice cream equal to the price plus half the distance travelled. What is the new equilibrium? In this new equilibrium, do the ice cream stands make more or less money?*

The new indifferent meter d' solves $p_A + d'/2 = p_B + (500 - d')/2$ which gives $d' = p_B - p_A + 250$. Profit functions are

$$u_A(p_A, p_B) = p_A \cdot d' = p_A \cdot (p_B - p_A + 250) = -p_A(p_A - (250 + p_B))$$

and

$$u_B(p_A, p_B) = p_B \cdot d = p_B \cdot (p_A - p_B + 250) = -p_B(p_B - (250 + p_A)).$$

Thus, stand A's best response to p_B is $125 + p_B/2$ and stand B's best response to p_A is $125 + p_A/2$. For each stand to be best responding to the other, we need $p_A = 250, p_B = 250.$
In both this case and the previous case, the indifferent meter turns out to be

$$d = \frac{500 - 500 + 500}{2} = 250$$

or

$$d' = \frac{250 - 250 + 500}{2} = 250.$$

As such, the number of ice creams sold by each vendor is unchanged. The only difference is vendors charge less than before, so they make less money.

- 6 points: Found Nash equilibrium
- 2 points: Found that profits decrease
- 2 (partial) points: Found new indifferent meter
- 2 (partial) points: Found best responses

4 Additional Notes

A good (free and online) course for learning game theory can be found here: https://www.coursera.org/learn/game-theory-1. Watson (2013) is a standard introudctory textbook in game theory. Fudenberg and Triole (1991) takes a much more mathematically rigorous approach and explores content at a deeper level.

The equation on the t-shirt is influenced by the Envelope Theorem, generalized by Milgrom and Segal (2002). It is used in modern-day analyses of auctions, as seen in Milgrom (2004).

5 References

Fudenberg, Drew, and Jean Tirole. *Game Theory*. Cambridge, Mass., Mit Press, 1991.

Milgrom, Paul, and Ilya Segal. “Envelope Theorems for Arbitrary Choice Sets.” *Econometrica*, vol. 70, no. 2, 2002.

Milgrom, Paul. *Putting Auction Theory to Work*. Cambridge, Cambridge Univ. Press, 2004.

Watson, Joel. *Strategy : An Introduction to Game Theory*. New York, W.W. Norton & Company, 2013.